A two-variable refinement of the Stark conjecture in the function field case

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Abstract

We propose a conjecture refining the Stark conjecture St(K/k,S) in the function field case. Of course St(K/k,S) in this case is a theorem due to Deligne and independently to Hayes. The novel feature of our conjecture is that it involves *two*- rather than one-variable algebraic functions over finite fields. Because the conjecture is formulated in the language and framework of Tate's thesis we have powerful standard techniques at our disposal to study it. We build a case for our conjecture by (i) proving a parallel result in the framework of adelic harmonic analysis which we dub the *adelic Stirling formula*, (ii) proving the conjecture in the genus zero case and (iii) explaining in detail how to deduce St(K/k,S) from our conjecture. In the genus zero case the class of two-variable algebraic functions thus arising includes all the *solitons* over a genus zero global field previously studied and applied by the author, collaborators and others. Ultimately the inspiration for this paper comes from striking examples given some years ago by R. Coleman and D. Thakur.

Contents

1. Introduction

We propose here a conjecture (Conjecture 9.5 below) refining the Stark conjecture St($K/k, S$) (Tate's formulation [Tate]) in the function field case, and we verify our conjecture in the genus zero case. Of course St($K/k, S$) in the function field case is a theorem due to Deligne [Tate] and independently to Hayes [Ha85]. The main novelty of our conjecture is that it involves *two*- rather than one-variable algebraic functions. The conjecture is formulated in the powerful language and framework of Tate's thesis. Ultimately this paper is inspired by remarkable examples of Coleman

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[Co88] and Thakur [Th91, §9.3]. To motivate the paper we devote most of this introduction to the discussion of an example similar to the ones originally considered by Coleman and by Thakur. We first discuss the example in "raw form" and then we explain how to look at it from the adelic point of view cultivated in this paper.

Consider the smooth projective model $C/\mathbb{F}_q$ of the affine plane curve

$$Y^q - Y = X^{q-1}$$

over $\mathbb{F}_q$. On the surface $C \times C$ (product formed over $\mathbb{F}_q$) consider the meromorphic function

$$\varphi = Y \otimes 1 - 1 \otimes Y - \frac{X \otimes 1}{1 \otimes X}.$$

Following the lead of Coleman, it can easily be verified that

$$\begin{gathered}\text{divisor of } \varphi \\ \equiv \text{ transpose of} \\ \text{graph}((X,Y) \mapsto (X^q, Y^q)) \\ + \sum_{\alpha \in \mathbb{F}_q^\times} \text{graph}((X,Y) \mapsto (\alpha X, \alpha + Y)), \\ \text{modulo horizontal and vertical divisors.}\end{gathered} \tag{1}$$

By the theory of correspondences on curves it follows that the analogously formed sum of endomorphisms of the Jacobian of C vanishes. Thus we reprove following Coleman—by a strikingly elementary argument—that the "universal Gauss sum"

$$-\sum_{\alpha \in \mathbb{F}_q^\times} ((X,Y) \mapsto (\alpha X, \alpha + Y)) \in \mathbb{Z}[\mathrm{Aut}(C/\mathbb{F}_q)]$$

acts on the Jacobian of C in the same way as does the q^{th} power Frobenius endomorphism.

Now view C as a covering of the T-line over $\mathbb{F}_q$ by setting

$$-T = Y^q - Y = X^{q-1}.$$

Still following Coleman, notice that by specializing the second variable to $\mathbb{F}_{q^d}$ in the example φ, we obtain a "Stark-like" function in the extension $\mathbb{F}_{q^d}(X,Y)/\mathbb{F}_q(T)$. We say "Stark-like" because the functions arising this way do not have precisely the right properties to qualify as Stark units in the sense of $\mathrm{St}(K/k,S)$; nonetheless, these functions would appear to be promising "raw material" for building Stark units since their divisors and Galois properties are evident. What is yet more intriguing is that this approach gives insight into the variation of the Stark unit as we vary the choice of completely split place. It is Coleman's brilliant idea to link two-variable algebraic functions to Stark's conjecture in this way.

We turn to follow Thakur's lead. We have

$$\varphi \left| \begin{array}{l} X \otimes 1 \mapsto \alpha X \\ Y \otimes 1 \mapsto \beta + Y \\ 1 \otimes X \mapsto X^{q^{N+1}} \\ 1 \otimes Y \mapsto Y^{q^{N+1}} \end{array} \right. = \prod_{\substack{a \in \mathbb{F}_q[T] \\ \deg a < N}} \frac{a + \beta T^N + T^{N+1} + \alpha/T}{a + T^N}, \tag{2}$$

for all $\alpha \in \mathbb{F}_q^\times$, $\beta \in \mathbb{F}_q$ and integers $N \geqslant 0$, as can be verified by manipulating Moore determinants.

By definition, the left side of the equation is the pull-back of φ via the map

$$((X,Y)\mapsto((\alpha X,\beta+Y),(X^{q^{N+1}},Y^{q^{N+1}}))):C\to C\times C.$$

The system of identities (2) uniquely characterizes our example φ, and illustrates an extremely valuable heuristic which we credit to Thakur, namely: "Frobenius interpolation formulas" with "Γ-partial-products" on the right side automatically have "functions of Coleman type" on the left side.

We pause to remark that a fairly extensive body of theory and applications of phenomena of Coleman/Thakur type has already been developed. For example, see [An92], [An94], [ABP04], [Si97a], [Si97b]; the applications to transcendence theory have been particularly notable. For a survey with a wealth of examples, see [Thak, Chaps. 4 and 8]. While the main emphasis in the theory to this point has been on the case of the basefield $\mathbb{F}_q(T)$, significant efforts to generalize the base have been made (see [An94] and [Thak]). But a general framework for understanding and classifying all these examples has been lacking. The main point of this paper is to rebuild the theory on the foundation of Tate's thesis so that a clear (albeit conjectural) picture of the situation over a general basefield emerges.

We return to the analysis of our example φ. We explain how to look at the example from the point of view peculiar to this paper. Put

$$k=\mathbb{F}_q(T),\quad K=\mathbb{F}_q(X,Y),\quad G=\mathrm{Gal}(K/k).$$

The group G is a copy of $\mathbb{F}_q^\times\times\mathbb{F}_q$ and in particular is abelian. Notice that φ is invariant under the diagonal action

$$(x\otimes y)^\sigma=x^\sigma\otimes y^\sigma$$

of G on $K\otimes_{\mathbb{F}_q}K$.

The example φ lives in a setting with somewhat subtle commutative algebra properties. Notice that (i) the ring extension

$$K\otimes_{\mathbb{F}_q}K/k\otimes_{\mathbb{F}_q}k$$

is a finite etale extension of Dedekind domains, (ii) the diagonal prime

$$\Delta_K=\ker(x\otimes y\mapsto xy):K\otimes_{\mathbb{F}_q}K\to K$$

lies above the diagonal prime

$$\Delta=\Delta_k=\ker(x\otimes y\mapsto xy):k\otimes_{\mathbb{F}_q}k\to k$$

and (iii) the prime of the diagonally G-invariant subring of $K\otimes_{\mathbb{F}_q}K$ below Δ_K has the same residue field as Δ, namely k. The upshot is that the diagonal completion of $k\otimes_{\mathbb{F}_q}k$ coincides with the diagonal completion of the diagonally G-invariant subring of $K\otimes_{\mathbb{F}_q}K$. We now simply identify the latter with the former. Under this identification the strange-looking formula

$$\varphi=\sum_{i=0}^\infty(T\otimes 1-1\otimes T)^{q^i}-\prod_{i=0}^\infty\left(\frac{T\otimes 1}{1\otimes T}\right)^{-q^i}.$$

makes sense.

Now let's put global class field theory into the picture. Let $\mathbb{A}$ be the adele ring of k and for each idele $a\in\mathbb{A}^\times$, let $\|a\|\in q^{\mathbb{Z}}$ be the corresponding idele norm. Let $\rho:\mathbb{A}^\times\to G$ be the reciprocity law homomorphism of global class field theory, redefined the modern way, as in Tate's article [Ta79], so that uniformizers map to geometric Frobenius elements and hence, for all c algebraic over $\mathbb{F}_q$ and $a\in\mathbb{A}^\times$, we have $c^{\rho(a)}=c^{\|a\|}$. (We follow this rule throughout the paper.) We make $\mathbb{A}^\times$ act on the perfect ring $\sqrt[q^\infty]{K}\otimes_{\mathbb{F}_q}\sqrt[q^\infty]{K}$ by the rule

$$(x\otimes y)^{(a)}=x^{\rho(a)}\otimes y^{\|a\|},$$

which makes sense since $\mathrm{Gal}(K/k) = \mathrm{Gal}(\sqrt[q^\infty]{K}/\sqrt[q^\infty]{k})$. Since the $\mathbb{A}^\times$-action defined above commutes with the diagonal action of G, we may naturally view all the functions

$$\varphi^{(a)} \quad (a \in \mathbb{A}^\times)$$

as elements of the perfection of the diagonal completion of $k \otimes_{\mathbb{F}_q} k$.

We need a few more definitions before getting to the "punchline". Let ord_Δ be the normalized additive valuation of $k \otimes_{\mathbb{F}_q} k$ giving the order of vanishing at the diagonal ideal, and let this additive valuation be extended to the perfection of the completion in evident fashion. Let $\mathcal{O} \subset \mathbb{A}$ be the maximal compact subring. Let ∞ be the unique place of k at which T has a pole. Let $\tau \in \mathbb{A}^\times$ be the idele whose component at ∞ is T^{-1} and whose components elsewhere are 1. Consider the $\mathbb{Z}$-valued Schwartz function $\Phi = \mathbf{1}_{\tau(T+1/T+\tau\mathcal{O})} - \mathbf{1}_{\tau(1+\tau\mathcal{O})}$ on $\mathbb{A}$, where $\mathbf{1}_S$ is probabilist's notation for the function taking the value 1 on the set S and 0 elsewhere.

We now get to the heart of the matter. The divisor formula (1) can be rewritten as the formula

$$\mathrm{ord}_\Delta\, \varphi^{(a)} = \sum_{x \in k} \Phi(a^{-1}x) \tag{3}$$

holding for all $a \in \mathbb{A}^\times$, plus a further assertion to the effect that no primes of the Dedekind domain $K \otimes_{\mathbb{F}_q} K$ other than those of the form

$$\ker\left((x \otimes y \mapsto x^{\rho(a)} y^{\|a\|}) : K \otimes_{\mathbb{F}_q} K \to \sqrt[q^\infty]{K}\right) \quad (a \in \mathbb{A}^\times)$$

enter into the prime factorization of φ in $K \otimes_{\mathbb{F}_q} K$. The interpolation formula (2) can be rewritten as the formula

$$\mathrm{ord}_\Delta \left(\varphi^{(a)} - 1 \otimes \prod_{x \in k^\times} x^{\Phi(a^{-1}x)} \right) > 0 \tag{4}$$

holding for all $a \in \mathbb{A}^\times$ such that $\|a\| > 1$. Formulas (3) and (4) very strongly suggest that there's more to the Coleman/Thakur phenomenon than just a few isolated examples. We now draw our discussion of the example φ to a close.

We turn to the task of describing our conjecture, in order at least to convey some its flavor. We continue with similar notation. Let k be any global field with constant field $\mathbb{F}_q$, let $\mathbb{A}$ be the adele ring of k, and let $\rho : \mathbb{A}^\times \to \mathrm{Gal}(k^{\mathrm{ab}}/k)$ be the reciprocity law homomorphism of global class field theory. We define a *Coleman unit* to be an element of the fraction field of the "big ring" $\sqrt[q^\infty]{k^{\mathrm{ab}}} \otimes_{\mathbb{F}_q^{\mathrm{ab}}} \sqrt[q^\infty]{k^{\mathrm{ab}}}$ which is (i) invariant under the diagonal action $(x \otimes y)^\sigma = x^\sigma \otimes y^\sigma$ of $\mathrm{Gal}(k^{\mathrm{ab}}/k)$ and (ii) a unit at every maximal ideal of the big ring not transformable under the action $(x \otimes y)^{(a)} = x^{\rho(a)} \otimes y^{\|a\|}$ of $\mathbb{A}^\times$ to the diagonal ideal $\ker(x \otimes y \mapsto xy)$. Our conjecture associates to every $\mathbb{Z}[1/q]$-valued Schwartz-function Φ on $\mathbb{A}$ such that $\Phi(0) = 0 = \hat{\Phi}(0)$ a Coleman unit φ for which analogues of (3) and (4) hold, the latter in a more general formulation (not subject to any restriction on $\|a\|$) relating the "leading Taylor coefficient" to a gadget we call the *Catalan symbol.* The values of the Catalan symbol are function field elements roughly analogous in structure to the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$. The Catalan symbol is defined in terms of a further gadget we call the *rational Fourier transform.* The characteristic feature of the theory of rational Fourier transforms is that the function $(x \mapsto \delta_{x0} - \delta_{x1}) : \mathbb{F}_q \to \{-1, 0, 1\}$ is assigned the role usually played by a nonconstant complex-valued character of $\mathbb{F}_q$.

As mentioned above, we prove our conjecture in genus zero. We believe that ideas from the author's papers [An94] and [An04] suitably combined have a fighting chance to prove our conjecture in general. Perhaps the rather different set of ideas from [An96] could also yield a proof. We hope for a proof of the conjecture in the general case not so heavily computational as the proof we give here in the genus zero case.

To make the point that our conjecture refines $\mathrm{St}(K/k,S)$, we provide a deduction of the latter from the former in §12 below. In fact a large fraction of the paper is devoted to developing the machinery needed to make the deduction of $\mathrm{St}(K/k,S)$ from our conjecture run smoothly. In particular, the technical tool we call the *adelic Stirling formula* (Theorem 7.7 below) is crucial for proving our "recipe" (Theorem 12.3 below) for the Stark unit conditional on our conjecture.

We conclude the introduction by remarking on the organization of the paper. Since we have provided a table of contents, we needn't make comments section-by-section. We just offer the following advice. After glancing at §2 to be apprised of general notation and terminology, the reader could very well start in §7, because that's where the main story-line of the paper begins; the preceding sections, which are more or less independent of each other, could be treated as references. But on the other hand, if the reader would take the trouble to work patiently through the sections before §7, he/she would be prepared to hear the main story undistracted by technical issues of an essentially nonarithmetical nature.

2. General notation and terminology

2.1

We denote the cardinality of a set S by $\#S$.

2.2

Given sets $S \subset X$, let $\mathbf{1}_S$ denote the real-valued function taking the value 1 on S and 0 on $X \setminus S$; we omit reference to X in the notation because it can always be inferred from context. We borrow this notation from the probabilists.

2.3

Rings are always commutative with unit. Let $A^\times$ denote the group of units of a ring A. Let $\mathbb{F}_q$ denote a field of $q < \infty$ elements.

2.4

A function v on a ring A taking values in $\mathbb{R} \cup \{+\infty\}$ is called an *additive valuation* if $v(a) = +\infty \Leftrightarrow a = 0$, $v(a+b) \geqslant \min(v(a), v(b))$ and $v(ab) = v(a) + v(b)$ for all $a, b \in A$. If A is a field and $v(A) = \mathbb{Z} \cup \{+\infty\}$, we say that v is *normalized.* Additive valuations are said to be *equivalent* if proportional. A *place* of a field k is an equivalence class of nontrivial additive valuations of k.

2.5

Given an integral domain A of characteristic $p > 0$, let A_{perf} be the closure of A under the extraction of p^{th} roots, i. e., the direct limit of the system $A \xrightarrow{x \mapsto x^p} A \xrightarrow{x \mapsto x^p} \dots$ of rings and homomorphisms. If $A = A_{\mathrm{perf}}$, then we say that A is *perfect.*

2.6

Given a locally compact totally disconnected topological space X, we denote by $\mathrm{Sch}(X)$ the *Schwartz space* of locally constant compactly supported complex-valued functions on X. (In the literature this is sometimes called instead the *Schwartz-Bruhat space*, cf. [RV].)

2.7 Moore determinants

Given an $\mathbb{F}_q$-algebra A and ring elements $x_1, \ldots, x_n \in A$, we define the *Moore determinant*

$$\mathrm{Moore}(x_1, \ldots, x_n) = \left| \begin{array}{cccc} x_1^{q^{n-1}} & x_1^{q^{n-2}} & \ldots & x_1 \\ \vdots & \vdots & & \vdots \\ x_n^{q^{n-1}} & x_n^{q^{n-2}} & \ldots & x_n \end{array} \right| = \det_{i,j=1}^{n} x_i^{q^{n-j}} \in A.$$

For this we have the well-known *Moore identity* (see [Goss, Chap. 1])

$$\mathrm{Moore}(x_1, \ldots, x_n) = \prod_{\ell=1}^{n} \sum_{c_{\ell+1} \in \mathbb{F}_q} \cdots \sum_{c_n \in \mathbb{F}_q} \left(x_\ell + \sum_{i=\ell+1}^{n} c_i x_i \right). \tag{5}$$

For example, working over the field of rational functions in a variable T with coefficients in $\mathbb{F}_q$, we have

$$\prod_{\substack{a \in \mathbb{F}_q[T] \\ \deg a = N \\ a:\ \text{monic in } T}} a = \frac{\mathrm{Moore}(T^N, \ldots, 1)}{\mathrm{Moore}(T^{N-1}, \ldots, 1)} = \prod_{i=0}^{N-1} (T^{q^N} - T^{q^i}) \tag{6}$$

for all nonnegative integers N, where $\deg a$ denotes the degree of a as a polynomial in T. The last equality is justified by the Vandermonde identity.

2.8 Ore determinants

The following variant of the Moore determinant will also be needed. Given an $\mathbb{F}_q$-algebra A, ring elements $x_1, \ldots, x_n \in A$, and an $\mathbb{F}_q$-linear functional ξ defined on the $\mathbb{F}_q$-span of $x_1, \ldots, x_n$, put

$$\mathrm{Ore}(\xi, x_1, \ldots, x_n) = \left| \begin{array}{cccc} \xi x_1 & x_1^{q^{n-2}} & \ldots & x_1 \\ \vdots & \vdots & & \vdots \\ \xi x_n & x_n^{q^{n-2}} & \ldots & x_n \end{array} \right| \in A.$$

We call $\mathrm{Ore}(\xi, x_1, \ldots, x_n)$ the *Ore determinant* of $\xi, x_1, \ldots, x_n$. For this we have a variant of the Moore identity which, for simplicity, we state under some special assumptions always fulfilled in practice. We suppose now that A is a field, and that $x_1, \ldots, x_n$ are $\mathbb{F}_q$-linearly independent, in which case $\mathrm{Moore}(x_1, \ldots, x_n) \neq 0$. Let V be the $\mathbb{F}_q$-linear span of $x_1, \ldots, x_n$. We suppose further that $\xi V \neq 0$. Then, so we claim,

$$\frac{\mathrm{Ore}(\xi, x_1, \ldots, x_n)}{\mathrm{Moore}(x_1, \ldots, x_n)} = \prod_{\substack{v \in V \\ \xi v = 1}} v^{-1}. \tag{7}$$

We call this relation the *Ore identity*. To prove the claim, first note that the ratio on the left side does not change if we replace $x_1, \ldots, x_n$ by any $\mathbb{F}_q$-basis of V. We may therefore assume without loss of generality that $\xi x_i = \delta_{i1}$ for $i = 1, \ldots, n$, in which case the claim follows from the Moore identity (5). The claim is proved. Relation (7) is a convenient way to restate what for our purposes is the main point of Ore-Elkies-Poonen duality (see [Goss, §4.14]).

3. The rational Fourier transform and the Catalan symbol ("toy" versions)

We discuss in "toy form" some simple algebraic notions later to be applied in the adelic context.

3.1 Notation

We fix vector spaces V and V^* over $\mathbb{F}_q$ of the same finite dimension. We fix a perfect $\mathbb{F}_q$-bilinear pairing

$$\langle \cdot, \cdot \rangle : V \times V^* \to \mathbb{F}_q.$$

We fix a nonconstant homomorphism

$$\lambda : \mathbb{F}_q \to U(1),$$

where $U(1)$ is the group of complex numbers of absolute value 1. We also fix a field K containing a copy of $\mathbb{F}_q$.

3.2 Toy Fourier transforms

Let $\mathrm{Sch}(V)$ be the set of $\mathbb{C}$-valued functions on V. Given $\Phi \in \mathrm{Sch}(V)$, put

$$\hat{\Phi}(v^*) = \mathcal{F}[\Phi](v^*) = \sum_{v \in V} \Phi(v)\lambda(-\langle v, v^* \rangle)$$

for all $v^* \in V^*$, thus defining the *toy Fourier transform*

$$\hat{\Phi} = \mathcal{F}[\Phi] \in \mathrm{Sch}(V^*).$$

Perhaps it is wasteful to define two notations for the Fourier transform, but it is quite convenient, and we follow this practice throughout the paper for all the versions of the Fourier transform that we consider.

For example, given $v_0 \in V$ and an $\mathbb{F}_q$-subspace $W \subset V$, we have

$$\mathcal{F}[\mathbf{1}_{v_0+W}](v^*) = \#W \cdot \mathbf{1}_{W^\perp}(v^*) \cdot \lambda(-\langle v_0, v^* \rangle) \tag{8}$$

for all $v^* \in V^*$, where $W^\perp$ is the subspace of V^* annihilated by W. Further, we have

$$\Phi(v) = (\#V)^{-1} \sum_{v^* \in V^*} \hat{\Phi}(v^*)\lambda(\langle v, v^* \rangle)$$

for all $\Phi \in \mathrm{Sch}(V)$ and $v \in V$; thus we can invert the toy Fourier transform explicitly. So far all this is completely familiar.

3.3 The function λ_0

We begin to warp things a bit. Put

$$\lambda_0 = \left(x \mapsto \begin{cases} 1 & \text{if } x = 0 \\ -1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 0, 1 \end{cases} \right) : \mathbb{F}_q \to \{-1, 0, 1\} \subset \mathbb{C}.$$

Then we have trivial identities

$$\begin{array}{rcl} \lambda_0(x) & = & q^{-1} \sum_{c \in \mathbb{F}_q^\times} (1 - \lambda(c))\lambda(-cx), \\ \lambda(-x) & = & -\sum_{c \in \mathbb{F}_q^\times} \lambda(-c)\lambda_0(c^{-1}x), \\ \sum_{c \in \mathbb{F}_q^\times} \lambda(cx) & = & \sum_{c \in \mathbb{F}_q^\times} \lambda_0(cx), \\ -\sum_{c \in \mathbb{F}_q^\times} \lambda(-c)\lambda(cx) & = & q\lambda_0(x) - \sum_{c \in \mathbb{F}_q^\times} \lambda_0(cx) \end{array} \tag{9}$$

holding for all $x \in \mathbb{F}_q$. Roughly speaking, these identities say that although λ_0 is not a character of $\mathbb{F}_q$ (save in the case $q = 2$), it behaves sufficiently like a character that we can base an alternative theory of Fourier transforms on it.

3.4 Toy rational Fourier transforms

Given $\Phi \in \mathrm{Sch}(V)$, put

$$\tilde{\Phi}(v^*) = \mathcal{F}_0[\Phi](v^*) = \sum_{v \in V} \Phi(v)\lambda_0(\langle v, v^* \rangle)$$

for all $v^* \in V^*$, thus defining the *toy rational Fourier transform*

$$\tilde{\Phi} = \mathcal{F}_0[\Phi] \in \mathrm{Sch}(V^*).$$

Via (9) we have

$$\begin{array}{rcl} \hat{\Phi}(v^*) & = & -\sum_{c \in \mathbb{F}_q^\times} \lambda(-c)\tilde{\Phi}(c^{-1}v^*), \\ \tilde{\Phi}(v^*) & = & q^{-1}\sum_{c \in \mathbb{F}_q^\times}(1-\lambda(c))\hat{\Phi}(cv^*) \end{array} \tag{10}$$

for all $v^* \in V$; thus we can express toy Fourier transforms in terms of their rational analogues and *vice versa*. For example, given $v_0 \in V$ and an $\mathbb{F}_q$-subspace $W \subset V$, we have

$$\mathcal{F}_0[\mathbf{1}_{v_0+W}] = \#W \cdot \left(\mathbf{1}_{W^\perp \cap [\langle v_0, \cdot \rangle = 0]} - \mathbf{1}_{W^\perp \cap [\langle v_0, \cdot \rangle = 1]}\right) \tag{11}$$

by combining (8), (9) and (10). The notation $[\langle v_0, \cdot \rangle = 1]$ is probabilist-style shorthand for $\{v^* \in V^* | \langle v_0, v^* \rangle = 1\}$. We use similar notation below without further comment. By combining the toy Fourier inversion formula with (9) and (10) we have

$$\begin{array}{rcl} \#V \cdot \Phi(v) & = & \sum_{v^* \in V} \lambda(\langle v, v^* \rangle)\left(-\sum_{c \in \mathbb{F}_q^\times} \lambda(-c)\tilde{\Phi}(c^{-1}v^*)\right) \\ & = & \sum_{v^* \in V^*}\left(-\sum_{c \in \mathbb{F}_q^\times} \lambda(-c)\lambda(c\langle v, v^* \rangle)\right)\tilde{\Phi}(v^*) \\ & = & \sum_{v^* \in V^*}\left(q\lambda_0(\langle v, v^* \rangle) - \sum_{c \in \mathbb{F}_q^\times} \lambda_0(c\langle v, v^* \rangle)\right)\tilde{\Phi}(v^*) \end{array}$$

for all $\Phi \in \mathrm{Sch}(V)$ and $v \in V$; thus we can invert the toy rational Fourier transform explicitly.

Let $\mathbb{Z}[1/q]$ be the ring obtained from $\mathbb{Z}$ by inverting q. Let $\mathrm{Sch}_0(V)$ be the group of $\mathbb{Z}[1/q]$-valued functions on V. Notice that we have $\mathcal{F}_0[\mathrm{Sch}_0(V)] = \mathrm{Sch}_0(V^*)$. It is because of the latter relation that $\mathcal{F}_0$ deserves to be called rational.

3.5 Toy Catalan symbols

Recall that K is a field containing $\mathbb{F}_q$. Let $\alpha : V \to K$ be an injective $\mathbb{F}_q$-linear map. For all $\Phi \in \mathrm{Sch}_0(V)$, put

$$\left(\begin{array}{c} \alpha \\ \Phi \end{array}\right) = \prod_{0 \neq v \in V} \alpha(v)^{\Phi(v)} \in K_{\mathrm{perf}}^\times.$$

The definition makes sense because $K_{\mathrm{perf}}^\times$ is a uniquely q-divisible group. We call $(\dot{\cdot})$ the *toy Catalan symbol*. It is clear that $\left(\begin{array}{c} \alpha \\ \Phi \end{array}\right)$ depends $\mathbb{Z}[1/q]$-linearly on Φ. We see the values of the toy Catalan symbol as being very roughly analogous in structure to the *Catalan numbers*

$$\frac{1}{n+1}\left(\begin{array}{c} 2n \\ n \end{array}\right) = \frac{(2n)!}{(n+1)!n!} = \prod_{0 \neq x \in \mathbb{Z}} x^{(\mathbf{1}_{(0,2n]} - \mathbf{1}_{(0,n+1]} - \mathbf{1}_{(0,n]})(x)},$$

whence the terminology.

Proposition 3.6. *Let $\alpha : V \to K$ be an injective $\mathbb{F}_q$-linear map. Let $W \subset V$ be an $\mathbb{F}_q$-subspace. Then the map*

$$\left(v \mapsto \begin{cases} \left(\begin{array}{c} \alpha \\ \mathbf{1}_{v+W} \end{array}\right) & \text{if } v \notin W \\ 0 & \text{if } v \in W \end{cases}\right) : V \to K$$

is $\mathbb{F}_q$-linear.

Proof. Choose a basis $w_1,\dots,w_n$ for W over $\mathbb{F}_q$. The $\mathbb{F}_q$-linear map

$$\left(v\mapsto\frac{\mathrm{Moore}(\alpha(v),\alpha(w_1),\dots,\alpha(w_n))}{\mathrm{Moore}(\alpha(w_1),\dots,\alpha(w_n))}\right):V\to K$$

coincides with the map in question. □

Lemma 3.7. *Let $\alpha:V\to K$ be an injective $\mathbb{F}_q$-linear map. Let $W\subset V$ be an $\mathbb{F}_q$-subspace. Then the map*

$$\left(v^*\mapsto\begin{cases}\left(\begin{array}{c}\alpha\\ -\mathbf{1}_{W\cap[\langle\cdot,v^*\rangle=1]}\end{array}\right) & \text{if } v^*\notin W^\perp\\ 0 & \text{if } v^*\in W^\perp\end{cases}\right):V^*\to K$$

is $\mathbb{F}_q$-linear.

Proof. We may assume without loss of generality that $W=V$. Choose an $\mathbb{F}_q$-basis $v_1,\dots,v_n\in V$. The $\mathbb{F}_q$-linear map

$$\left(v^*\mapsto\frac{\mathrm{Ore}(v^*\circ\alpha^{-1},\alpha(v_1),\dots,\alpha(v_n))}{\mathrm{Moore}(\alpha(v_1),\dots,\alpha(v_n))}\right):V^*\to K$$

coincides with the map in question. □

Proposition 3.8. *Let $\beta:V^*\to K$ be an injective $\mathbb{F}_q$-linear map. Let $W\subset V$ be an $\mathbb{F}_q$-subspace. Fix $v_0\in V\setminus W$. Then the map*

$$\left(v\mapsto\begin{cases}\left(\begin{array}{c}\beta\\ \mathcal{F}_0[\mathbf{1}_{v+W}-\mathbf{1}_{v_0+W}]\end{array}\right) & \text{if } v\notin W\\ 0 & \text{if } v\in W\end{cases}\right):V\to K$$

is $\mathbb{F}_q$-linear.

This delicate property of the toy rational Fourier transform is the main formal justification for defining it as we have. In the adelic context there will be a similar phenomenon (see Theorem 8.5 below).

Proof. For $v\in V\setminus W$ we have by (11) and the definitions that

$$\begin{aligned}
&\left(\begin{array}{c}\beta\\ \mathcal{F}_0[\mathbf{1}_{v+W}-\mathbf{1}_{v_0+W}]\end{array}\right)^{1/\#W}\\
&=\left(\begin{array}{c}\beta\\ \mathbf{1}_{W^\perp\cap[\langle v,\cdot\rangle=0]}-\mathbf{1}_{W^\perp\cap[\langle v,\cdot\rangle=1]}-\mathbf{1}_{W^\perp\cap[\langle v_0,\cdot\rangle=0]}+\mathbf{1}_{W^\perp\cap[\langle v_0,\cdot\rangle=1]}\end{array}\right)\\
&\stackrel{\star}{=}\left(\begin{array}{c}\beta\\ -\mathbf{1}_{W^\perp\cap[\langle v,\cdot\rangle\neq0]}-\mathbf{1}_{W^\perp\cap[\langle v,\cdot\rangle=1]}+\mathbf{1}_{W^\perp\cap[\langle v_0,\cdot\rangle\neq0]}+\mathbf{1}_{W^\perp\cap[\langle v_0,\cdot\rangle=1]}\end{array}\right)\\
&=\left(\begin{array}{c}\beta\\ -\mathbf{1}_{W^\perp\cap[\langle v,\cdot\rangle=1]}+\mathbf{1}_{W^\perp\cap[\langle v_0,\cdot\rangle=1]}\end{array}\right)^q,
\end{aligned}$$

which by the preceding lemma proves the result. □

3.9 Remark

The star marks the main trick of the toy theory. It is nothing more than the fact that for any three subsets A, B and C of a set X we have $\mathbf{1}_{A\cap B} - \mathbf{1}_{A\cap C} = -\mathbf{1}_{A\setminus B} + \mathbf{1}_{A\setminus C}$. We shall use the trick again to make explicit calculations in §10.

4. A functor and related identities

We work out identities in which on the left side appear functions on a certain commutative affine group scheme over $\mathbb{F}_q$ evaluated at certain special points, and in which on the right side appear expressions interpretable as values of naturally occurring toy Catalan symbols. These identities are used in the sequel for no purpose other than as an inputs to the proof in §10 of the genus zero case of the conjecture to which we allude in the title of the paper. Some similar identities were studied and applied in the papers [An92], [An94] and [ABP04], but there are two novel features here. Namely, (i) we work out connections with duality and (ii) we allow for the possibility of infinite ramification at infinity.

4.1 The functor $\mathcal{H}$ and associated structures

4.1.1 Let t be a variable. Given an $\mathbb{F}_q$-algebra R, let $R((1/t))$ be the ring consisting of all power series $\sum_i a_i t^i$ with coefficients $a_i \in R$ vanishing for $i \gg 0$, and put $\operatorname{Res} t = \infty \left(\sum_i a_i t^i dt\right) = -a_{-1} \in R$. Let $R[t] \subset R((1/t))$ (resp., $R[[1/t]] \subset R((1/t))$) be the subring consisting of power series $\sum_i a_i t^i$ such that $a_i = 0$ unless $i \geqslant 0$ (resp., $i \leqslant 0$). When R is a field, we denote by $R(t)$ the field of rational functions in t with coefficients in R, and we identify $R((1/t))$ with the completion of $R(t)$ at the infinite place.

4.1.2 We define a representable group-valued functor $\mathcal{H}$ of $\mathbb{F}_q$-algebras R by the rule

$$\mathcal{H}(R) = \varprojlim (R[t]/m(t))^\times \times (1 + (1/t)R[[1/t]]),$$

where the inverse limit is extended over monic $m(t) \in \mathbb{F}_q[t]$ ordered by the divisibility relation, and the group law is induced by multiplication. The commutative affine group scheme $\mathcal{H}$ is the natural one to consider in connection with the problem of constructing the maximal abelian extension of a field of rational functions in one variable with coefficients in $\mathbb{F}_q$. Later, in §10, we discuss and apply the arithmetical properties of $\mathcal{H}$. But now we focus on properties more in the line with classical symmetric function theory.

4.1.3 By definition, to give an R-valued point $P \in \mathcal{H}(R)$ is to give a power series

$$P_\infty(t) \in 1 + (1/t)R[[1/t]]$$

and for each monic $m = m(t) \in \mathbb{F}_q[t]$ a congruence class

$$P_m(t) \bmod m(t) \in (R[t]/m(t))^\times,$$

subject to the constraints that

$$m|n \Rightarrow P_m(t) \equiv P_n(t) \bmod m(t)$$

for all monic $m, n \in \mathbb{F}_q[t]$. For definiteness, we always choose the representative $P_m(t) \in R[t]$ to be of least possible degree in its congruence class modulo $m(t)$.

4.1.4 We produce useful examples of R-valued points of $\mathcal{H}$ as follows. Let $M(t) \in R[t]$ be a monic polynomial of degree d such that for all monic $m(t) \in \mathbb{F}_q[t]$, the resultant of $M(t)$ and $m(t)$

is a unit of R. We then define $[M(t)] \in \mathcal{H}(R)$ by setting $[M(t)]_\infty = M(t)/t^d$ and

$$[M(t)]_m = \text{remainder of } M(t) \text{ upon division by } m(t)$$

for each monic $m = m(t) \in \mathbb{F}_q[t]$. Given for $i = 1, 2$ a polynomial $M_i(t) \in R[t]$ for which $[M_i(t)] \in \mathcal{H}(R)$ is defined, note that $[M_1(t)][M_2(t)] = [M_1(t)M_2(t)]$.

4.1.5 Given $x = x(t) \in \mathbb{F}_q(t)$ and $P \in \mathcal{H}(R)$, put

$$\theta_x(P) = \mathrm{Res}_{t=\infty}\, x(t)(P_m(t) - P_\infty(t)/t)\, dt \in R \tag{12}$$

where $m = m(t) \in \mathbb{F}_q[t]$ is any monic polynomial such that $mx \in \mathbb{F}_q[t]$. It is easy to verify that the right side of (12) is independent of m and hence $\theta_x(P)$ is well-defined. Clearly, $\theta_x(P)$ depends $\mathbb{F}_q$-linearly on x. Now write

$$x = \langle x \rangle + \lfloor x \rfloor \quad (\langle x \rangle \in (1/t)\mathbb{F}_q[[1/t]],\ \ \lfloor x \rfloor \in \mathbb{F}_q[t]).$$

As the notation suggests, we think of $\langle x \rangle$ as the "fractional part" and $\lfloor x \rfloor$ as the "integer part" of x. With P, x and m as above, we have

$$\theta_x(P) = \mathrm{Res}_{t=\infty}(\langle x \rangle(t) P_m(t) - \lfloor x \rfloor(t) P_\infty(t)/t)\, dt \tag{13}$$

after throwing away terms which do not contribute to the residue. Note that the right side of (13) remains unchanged if we replaced $P_m(t)$ by any member of its congruence class in $R[t]$ modulo $m(t)$. If moreover $P = [M(t)]$, where $M(t) \in R[t]$ is a monic polynomial of degree d whose resultant with each monic element of $\mathbb{F}_q[t]$ is a unit of R, then we have

$$\theta_x([M(t)]) = \mathrm{Res}_{t=\infty}(\langle x \rangle(t) - \lfloor x \rfloor(t) t^{-d-1}) M(t) dt, \tag{14}$$

after replacing $P_m(t) = [M(t)]_m$ in (13) by $M(t)$.

In the next three propositions we study the values of the natural transformations θ_x at certain special points of $\mathcal{H}$.

Proposition 4.2. *Let T be a variable independent of t. Let N be a nonnegative integer. Fix nonzero $x \in \mathbb{F}_q(t)$ and put*

$$X = X(t) = \langle x \rangle + t^N \lfloor x \rfloor \in \mathbb{F}_q(t)^\times.$$

Consider the point

$$P = \left[\prod_{i=0}^{N} (t - T^{q^i})\right]^{-1} \in \mathcal{H}(\mathbb{F}_q(T)).$$

Then we have

$$\theta_x(P) = \prod_{\substack{a \in \mathbb{F}_q[T] \\ \deg a < N}} \frac{X(T) + a}{T^N + a} \in \mathbb{F}_q(T)^\times. \tag{15}$$

Proof. To simplify writing, put $K = \mathbb{F}_q(T)$ and $T_i = T^{q^i}$. Let $X_0, \ldots, X_N$ be independent variables, independent also of t and T, and put

$$F(X_0, \ldots, X_N) = \frac{\left|\begin{matrix} X_N & \ldots & X_0 \\ T_N^{N-1} & \ldots & T_0^{N-1} \\ \vdots & & \vdots \\ 1 & \ldots & 1 \end{matrix}\right|}{\left|\begin{matrix} T_N^N & \ldots & T_0^N \\ \vdots & & \vdots \\ 1 & \ldots & 1 \end{matrix}\right|} \in K[X_0, \ldots, X_N].$$

Choose monic $m = m(t) \in \mathbb{F}_q[t]$ such that $mx \in \mathbb{F}_q[t]$ and write

$$\prod_{i=0}^{N}(t - T_i) \cdot P_m(t) = 1 - W(t)m(t) \quad (W(t) \in K[t]).$$

Then we have, so we claim,

$$\begin{aligned} & \theta_x(P) \\ = \ & \mathrm{Res}_{t=\infty}\left(\langle x\rangle(t)\left(\frac{1 - W(t)m(t)}{\prod_{i=0}^{N}(t - T_i)}\right) - \lfloor x\rfloor(t)\, t^{-1}\prod_{i=0}^{N}(1 - T_i/t)^{-1}\right) dt \\ = \ & -\mathrm{Res}_{t=\infty}\left(\frac{\langle x\rangle(t)W(t)m(t) + t^N\lfloor x\rfloor(t)}{\prod_{i=0}^{N}(t - T_i)}dt\right) \\ = \ & F(X(T_0), \ldots, X(T_N)) \\ = \ & \frac{\mathrm{Moore}(X(T), T^{N-1}, \ldots, 1)}{\mathrm{Moore}(T^N, \ldots, 1)}. \end{aligned}$$

This chain of equalities is justified as follows. We get the first equality by plugging into version (13) of the definition of θ. We get the second equality after throwing away the term $\frac{\langle x\rangle(t)}{\prod_{i=0}^{N}(t-T_i)}dt$, which does not contribute to the residue, and then rearranging in evident fashion. Since we have

$$\frac{1}{\prod_{i=0}^{N}(t - T_i)} = F\left(\frac{1}{t - T_0}, \ldots, \frac{1}{t - T_N}\right),$$

we get the penultimate equality by "sum of residues equals zero" for meromorphic differentials on the t-line over K. The last equality follows directly from the definitions. The claim is proved. Via the Moore determinant identity (5), the result follows. □

Proposition 4.3. *Let T and x be as in the preceding proposition. Let ν be a nonnegative integer. Put*

$$X = X(t) = \langle x\rangle + \lfloor t^{-\nu-1}x\rfloor \in \mathbb{F}_q(t).$$

Consider the point

$$P = \left[\prod_{i=0}^{\nu-1}(t - T^{q^i})\right] \in \mathcal{H}(\mathbb{F}_q(T)).$$

Consider the $\mathbb{F}_q$-linear functionals

$$\left.\begin{array}{rcr} \xi & = & (a(T) \mapsto \mathrm{Res}_{t=\infty}(\langle x\rangle(t) - t^{-\nu-1}\lfloor x\rfloor(t))\, a(t)\, dt) \\ \xi_1 & = & (a(T) \mapsto -\mathrm{Res}_{t=\infty}\, t^{-\nu-1}\, a(t)\, dt) \end{array}\right\} : V \to \mathbb{F}_q,$$

where

$$V = (\mathbb{F}_q\text{-span of } 1, T, \ldots, T^\nu) \subset \mathbb{F}_q(T).$$

Then we have

$$\theta_x(P) = \begin{cases} \displaystyle\prod_{\substack{a\in V \\ \xi_1 a = 1}} a \Big/ \prod_{\substack{a\in V \\ \xi a = 1}} a \in \mathbb{F}_q(T)^\times & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0. \end{cases} \tag{16}$$

Moreover, we have

$$\theta_x(P) = 0 \Leftrightarrow \langle x\rangle - t^{-\nu-1}\lfloor x\rfloor \in t^{-\nu-2}\mathbb{F}_q[[1/t]] + \mathbb{F}_q[t] \tag{17}$$

and

$$\theta_x(P) = 0 \Rightarrow X \neq 0. \tag{18}$$

Proof. Write

$$\prod_{i=0}^{\nu-1}(t - T^{q^i}) = \sum_{i=0}^{\nu}(-1)^i e_i t^{\nu-i}.$$

Then e_i for $i = 1, \ldots, \nu$ is the i^{th} elementary symmetric function of $T, T^q, \ldots, T^{q^{\nu-1}}$, and $e_0 = 1$. By specializing the well-known representation of the i^{th} elementary symmetric function as a quotient of determinants (see [Mac, I,3]) to the present case we have

$$e_i = \frac{\mathrm{Moore}(T^\nu, \ldots, \widehat{T^{\nu-i}}, \ldots, 1)}{\mathrm{Moore}(T^{\nu-1}, \ldots, 1)}$$

for $i = 0, \ldots, \nu$; here the term bearing the "hat" is to be omitted, and if $\nu = 0$, the denominator (an empty determinant) is put to 1. By means of the Ore identity (7) we can express the right side of the claimed identity as the ratio of Ore determinants

$$\frac{\mathrm{Ore}(\xi, T^\nu, \ldots, 1)}{\mathrm{Ore}(\xi_1, T^\nu, \ldots, 1)},$$

and the latter (note that $\xi_1(T^i) = \delta_{i\nu}$ for $i = 0, \ldots, \nu$) can be brought to the form

$$\sum_{i=0}^{\nu}(-1)^i e_i \xi(T^{\nu-i})$$

by straightforward manipulation of determinants. But this last equals $\theta_x(P)$ by (14). Thus (16) is proved. It follows that $\xi = 0 \Leftrightarrow \theta_x(P) = 0$. Moreover, $\xi = 0$ if and only if the condition on the right side of (17) is fulfilled. Therefore (17) holds.

We turn, finally, to the proof of (18). Consider the Laurent expansion $\sum_i b_i t^i$ of x at $t = \infty$, where $b_i \in \mathbb{F}_q$ and $b_i = 0$ for $i \gg 0$. Since $x \neq 0$, not all the coefficients b_i vanish. Moreover, by (17), we have $b_{i-\nu-1} = b_i$ for $i = 0, \ldots, \nu$. Finally, by definition of X, the Laurent expansion of X at $t = \infty$ is $\sum_{i<0} b_i t^i + \sum_{i\geqslant 0} b_{i+\nu+1} t^i$. The latter Laurent expansion does not vanish identically since to form it we have merely suppressed some repetitions of digits in the Laurent expansion $\sum_i b_i t^i$. Thus relation (18) holds. □

PROPOSITION 4.4. *Let T, x, ν, X and P be as in the preceding proposition. Assume now that $\theta_x(P) = 0$ (and hence $X \neq 0$). Put*

$$\varpi = T \otimes 1 - 1 \otimes T \in \mathbb{F}_q(T) \otimes_{\mathbb{F}_q} \mathbb{F}_q(T).$$

(Note that the ring on the right is a principal ideal domain of which ϖ is a prime element.) Consider the point

$$\tilde{P} = \left[t - T^{q^\nu} \otimes 1\right]^{-1} \left[\prod_{i=0}^{\nu}(t - 1 \otimes T^{q^i})\right] \in \mathcal{H}(\mathbb{F}_q(T) \otimes_{\mathbb{F}_q} \mathbb{F}_q(T))$$

(which reduces modulo ϖ to P). We have

$$\theta_x(\tilde{P}) \equiv (1 \otimes C) \cdot \varpi^{q^\nu} \bmod \varpi^{q^\nu+1}, \tag{19}$$

where

$$C = X(T^{q^\nu}) \prod_{i=0}^{\nu-1}(T^{q^\nu} - T^{q^i}) \in \mathbb{F}_q(T)^\times.$$

Proof. Put $K = \mathbb{F}_q(T)$ to simplify writing. Fix monic $m = m(t) \in \mathbb{F}_q[t]$ such that $mx \in \mathbb{F}_q[t]$. Put

$$Q = [t - T^{q^\nu}]^{-1} \in \mathcal{H}(K),$$

noting that

$$Q_\infty(t) = \left(1 - T^{q^\nu}/t\right)^{-1} \in 1 + (1/t)K[[1/t]],$$

and write

$$(t - T^{q^\nu})Q_m(t) = 1 - W(t)m(t) \quad (W(t) \in K[t]).$$

Let

$$1 \otimes Q_m(t) \in (K \otimes_{\mathbb{F}_q} K)[t], \quad 1 \otimes Q_\infty(t) \in 1 + (1/t)(K \otimes_{\mathbb{F}_q} K)[[1/t]]$$

be the results of applying the homomorphism

$$(x \mapsto 1 \otimes x) : K \to K \otimes_{\mathbb{F}_q} K$$

coefficient by coefficient to $Q_m(t)$ and $Q_\infty(t)$, respectively. Then, so we claim, we have congruences

$$\tilde{P}_m(t) \equiv (1 + \varpi^{q^\nu}(1 \otimes Q_m(t))) \prod_{i=0}^{\nu-1} (t - 1 \otimes T^{q^i}) \bmod (m(t), \varpi^{q^\nu+1}),$$

$$\tilde{P}_\infty(t) \equiv \left(1 + \varpi^{q^\nu}(1 \otimes Q_\infty(t))/t\right) \prod_{i=0}^{\nu-1} (1 - (1 \otimes T^{q^i})/t) \bmod \varpi^{q^\nu+1}.$$

To verify the first congruence, multiply both sides by

$$t - T^{q^\nu} \otimes 1 = (t - 1 \otimes T^{q^\nu}) - \varpi^{q^\nu}$$

and see that both sides reduce to $\prod_{i=0}^{\nu}(t - 1 \otimes T^{q^i})$ modulo $(m(t), \varpi^{q^\nu+1})$. To verify the second congruence multiply both sides by

$$1 - \frac{T^{q^\nu} \otimes 1}{t} = 1 - \frac{1 \otimes T^{q^\nu}}{t} - \frac{\varpi^{q^\nu}}{t}$$

and see that both sides reduce to $\prod_{i=0}^{\nu}\left(1 - \frac{1 \otimes T^{q^i}}{t}\right)$ modulo $\varpi^{q^\nu+1}$. Thus the claim is proved.

Continuing our calculation, we now plug into version (13) of the definition of θ, taking into account our hypothesis that $\theta_x(P) = 0$. We find that

$$\begin{aligned} C &= \mathrm{Res}_{t=\infty} \left(\langle x \rangle(t) Q_m(t) \prod_{i=0}^{\nu-1}(t - T^{q^i}) \right. \\ &\qquad \left. - \lfloor x \rfloor(t) Q_\infty(t) t^{-2} \prod_{i=0}^{\nu-1}(1 - T^{q^i}/t) \right) dt \\ &= \mathrm{Res}_{t=\infty} \frac{-\langle x \rangle(t) W(t) m(t) + \langle x \rangle(t) - t^{-\nu-1}\lfloor x \rfloor(t)}{(t - T^{q^\nu})} \prod_{i=0}^{\nu-1}(t - T^{q^i}) dt, \end{aligned}$$

where $C \in K$ is the unique coefficient for which (19) holds. By (17) and another application of our hypothesis that $\theta_x(P) = 0$, we have

$$C = -\mathrm{Res}_{t=\infty} \frac{\langle x \rangle(t) W(t) m(t) + \lfloor t^{-\nu-1} x \rfloor(t)}{(t - T^{q^\nu})} \prod_{i=0}^{\nu-1}(t - T^{q^i}) dt.$$

Finally, we get the claimed value for C by applying "sum of residues equals zero" for meromorphic differentials on the t-line over K. $\square$

5. Two group-theoretical lemmas

We prove a couple of technical results which in the sequel are used for no purpose other than as inputs to the proof of Theorem 12.3. We "quarantine" the results here since they are general group-theoretical facts whose proofs would be a distraction from the main story.

Lemma 5.1. *Let $q > 1$ be an integer. Let Γ be an abelian group equipped with a nonconstant homomorphism $\|\cdot\| : \Gamma \to q^{\mathbb{Z}} \subset \mathbb{R}^\times$. Let $\mathbb{Z}[\Gamma]$ be the integral group ring of Γ and let $\mathcal{J} \subset \mathbb{Z}[\Gamma]$ be the kernel of the ring homomorphism $\mathbb{Z}[\Gamma] \to \mathbb{R}$ induced by $\|\cdot\|$. For every subgroup $\Gamma' \subset \Gamma$, let $\mathcal{I}(\Gamma') \subset \mathbb{Z}[\Gamma]$ be the ideal generated by differences of elements of Γ'. Let $\Pi \subset \Gamma$ be a subgroup of finite index and put $\Pi_1 = \Pi \cap \ker \|\cdot\|$. Then we have $\mathcal{I}(\Pi) \cap \mathcal{J} = \mathcal{I}(\Pi_1) + \mathcal{I}(\Pi) \cdot \mathcal{J}$.*

Proof. Since $\mathcal{I}(\Pi_1) \subset \mathcal{I}(\Pi) \cap \mathcal{J}$, we may pass to the quotient $\mathbb{Z}[\Gamma/\Pi_1] = \mathbb{Z}[\Gamma]/\mathcal{I}(\Pi_1)$ in order to carry out our analysis of ideals. After replacing Γ by Γ/Π_1, we may simply assume that $\Pi_1 = \{1\}$, in which case the function $\|\cdot\|$ maps Π isomorphically to a subgroup of $q^{\mathbb{Z}}$. We cannot have $\Pi = \{1\}$ lest $[\Gamma : \Pi] = \infty$, and hence Π is a free abelian group of rank 1. Let $\pi_0 \in \Pi$ be a generator. Then $\mathcal{I}(\Pi)$ is the principal ideal of $\mathbb{Z}[\Gamma]$ generated by $1 - \pi_0$. Suppose now that we are given $f \in \mathcal{I}(\Pi) \cap \mathcal{J}$. Then we can write $f = (1 - \pi_0)g$ for some $g \in \mathbb{Z}[\Gamma]$ and since $1 - \|\pi_0\| \neq 0$, we must have $g \in \mathcal{J}$. Therefore we have $f \in \mathcal{I}(\Pi) \cdot \mathcal{J}$. □

Lemma 5.2. *Let k be a field. Fix an algebraic closure $\bar{k}/k$. Let k^{ab} be the abelian closure of k in $\bar{k}$. Let K/k be a finite subextension of k^{ab}/k. Let $\mu(K)$ be the group of roots of unity in K and assume that $\#\mu(K) < \infty$. Put $G = \mathrm{Gal}(K/k)$ and let $J \subset \mathbb{Z}[G]$ be the ideal of the integral group ring annihilating $\mu(K)$. Let $\epsilon : \mathbb{Z}[G] \to \bar{k}^\times$ be a homomorphism of abelian groups such that $\epsilon|_J \in \mathrm{Hom}_G(J, K^\times)$. Then ϵ takes all its values in $(k^{\mathrm{ab}})^\times$.*

The lemma is a variant of [St80, Lemma 6] and [Tate, p. 83, Prop. 1.2].

Proof. Let k^{sep} be the separable algebraic closure of k in $\bar{k}$. Put $e = [\mathbb{Z}[G] : J] = \#\mu(K)$. Since $e\mathbb{Z}[G] \subset J$ and e is prime to the characteristic of $\bar{k}$, the homomorphism ϵ takes all its values in $(k^{\mathrm{sep}})^\times$, and moreover the restriction map

$$(\psi \mapsto \psi|_J) : \mathrm{Hom}(\mathbb{Z}[G], (k^{\mathrm{sep}})^\times) \to \mathrm{Hom}(J, (k^{\mathrm{sep}})^\times)$$

is surjective. Put $\mathfrak{g} = \mathrm{Gal}(k^{\mathrm{sep}}/k)$. Regard $\mathbb{Z}[G]$ and J as (left) $\mathfrak{g}$-modules by inflation. Given two (left) $\mathfrak{g}$-modules A and B, let the group $\mathrm{Hom}(A, B)$ of homomorphisms of abelian groups be equipped with (left) $\mathfrak{g}$-module structure by the rule $(\sigma h)(a) = \sigma(h(\sigma^{-1}a))$. Fix a generator $\zeta \in \mu(K)$. From the exact sequence of G-modules

$$0 \to J \to \mathbb{Z}[G] \xrightarrow{\zeta \mapsto \zeta^{\mathbf{a}}} \mu(K) \to 0$$

we deduce an exact sequence

$$0 \to \mathbb{Z}/e\mathbb{Z} \to \mathrm{Hom}(\mathbb{Z}[G], (k^{\mathrm{sep}})^\times) \to \mathrm{Hom}(J, (k^{\mathrm{sep}})^\times) \to 0$$

of $\mathfrak{g}$-modules, where we make the evident identification

$$\mathbb{Z}/e\mathbb{Z} = \mathrm{Hom}_G(\mu(K), (k^{\mathrm{sep}})^\times).$$

Since $\mathfrak{g}$ acts trivially on $\mathbb{Z}/e\mathbb{Z}$, we can extract an exact sequence

$$\mathrm{Hom}_{\mathfrak{g}}(\mathbb{Z}[G], (k^{\mathrm{sep}})^\times) \to \mathrm{Hom}_{\mathfrak{g}}(J, (k^{\mathrm{sep}})^\times) \xrightarrow{\delta} \mathrm{Hom}_{\text{loc. const.}}(\mathfrak{g}, \mathbb{Z}/e\mathbb{Z})$$

of abelian groups from the long exact sequence in Galois cohomology. The group on the right is the group of locally constant homomorphisms from $\mathfrak{g}$ to $\mathbb{Z}/e\mathbb{Z}$. When we make the boundary map δ explicit by a diagram-chase, we find that

$$\epsilon(\sigma^{-1}\mathbf{a})^\sigma = (\zeta^{\mathbf{a}})^{\delta[\epsilon|_J](\sigma)}\epsilon(\mathbf{a})$$

for all $\mathbf{a} \in \mathbb{Z}[G]$ and $\sigma \in \mathfrak{g}$. (In this nonabelian setting, even though we employ exponential notation, we remain steadfastly leftist—we follow the rule $(x^\tau)^\sigma = x^{\sigma\tau}$.) It follows that ϵ commutes with every $\sigma \in \mathrm{Gal}(k^{\mathrm{sep}}/k^{\mathrm{ab}})$, and since every such σ acts trivially on $\mathbb{Z}[G]$, it follows finally that ϵ takes all its values in $(k^{\mathrm{ab}})^\times$. □

6. The local Stirling formula

In this section we work in the setting of harmonic analysis on a nonarchimedean local field. The main result here is the *local Stirling formula* (Theorem 6.6), which is a technical result used in the sequel for no purpose other than as an input to the proof of the adelic Stirling formula (Theorem 7.7). To motivate the local Stirling formula, we prove a corollary (Corollary 6.11, which is not needed in the sequel) which begins to explain the relationship with the classical Stirling formula. See [RV] for background on local harmonic analysis.

6.1 Data

- Let k be a nonarchimedean local field. (It is not necessary to assume that k is of positive characteristic.)
- Let $\mathbf{e} : k \to U(1)$ be a nonconstant continuous homomorphism from the additive group of k to the group of complex numbers of absolute value 1.

All the constructions in §6 proceed naturally from these choices.

6.2 Notation

- Let $\mathcal{O}$ be the maximal compact subring of k.
- Fix $\kappa \in k^\times$ such that $\kappa^{-1}\mathcal{O} = \{\xi \in k \mid \xi\mathcal{O} \subseteq \ker \mathbf{e}\}$.
- Let μ be Haar measure on k, normalized by $\mu\mathcal{O} \cdot \mu(\kappa^{-1}\mathcal{O}) = 1$.
- Let $\mu^\times$ be Haar measure on $k^\times$, normalized by $\mu^\times\mathcal{O}^\times = 1$.
- Let q be the cardinality of the residue field of $\mathcal{O}$.
- For each $a \in k$, put $\|a\| = \frac{\mu(a\mathcal{O})}{\mu\mathcal{O}}$ and $\operatorname{ord} a = -\frac{\log \|a\|}{\log q}$.
- Let $\mathrm{Sch}(k)$ be the Schwartz space of functions on k.

6.3 Basic rules of calculation

To help reconcile the present system of notation to whatever system the reader might be familiar with, we recall a few facts routinely used below.

- $\int f(x)d\mu(x) = \|a\| \int f(ax)d\mu(x)$ for μ-integrable f and $a \in k^\times$.
- $\mu\mathcal{O} = \|\kappa\|^{1/2}$ and $\mu\mathcal{O}^\times = \frac{q-1}{q}\|\kappa\|^{1/2}$.
- $\|\cdot\|$ is an absolute value of k with respect to which k is complete.
- ord is a normalized additive valuation of k.
- $\int f(x)d\mu(x) = \frac{q-1}{q}\|\kappa\|^{1/2} \int f(t)\|t\|d\mu^\times(t)$ for μ-integrable f.

6.4 Fourier transforms

6.4.1 Given a complex-valued μ-integrable function f on k, put

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \int f(x)\mathbf{e}(-x\xi)d\mu(x)$$

for all $\xi \in k$, thus defining the *Fourier transform* $\hat{f}$, denoted also by $\mathcal{F}[f]$, which is again a complex-valued function on k. The Fourier transform $\hat{f}$ is continuous and tends to 0 at infinity. If f is compactly supported, then $\hat{f}$ is locally constant. If f is locally constant, then $\hat{f}$ is compactly supported. For example, we have

$$\mathcal{F}[\mathbf{1}_{x+b\mathcal{O}}](\xi) = \|\kappa\|^{1/2}\|b\|\mathbf{1}_{b^{-1}\kappa^{-1}\mathcal{O}}(\xi)\mathbf{e}(-x\xi) \tag{20}$$

for all $b \in k^\times$ and $x, \xi \in k$.

6.4.2 The Schwartz class $\mathrm{Sch}(k)$ is stable under Fourier transform and the *Fourier inversion formula* states that

$$\Phi(x) = \int \mathbf{e}(x\xi)\hat{\Phi}(\xi)d\mu(\xi)$$

for all $\Phi \in \mathrm{Sch}(k)$. Our normalization of μ is chosen to make the inversion formula hold in the latter particularly simple form; in general there would be a positive constant depending on μ but independent of Φ multiplying the right side. The Fourier inversion formula implies the *squaring rule*

$$\mathcal{F}^2[\Phi](x) = \Phi(-x) \tag{21}$$

for all $\Phi \in \mathrm{Sch}(k)$. We mention also the very frequently used *scaling rule*

$$\mathcal{F}[f^{(a)}] = \|a\|\hat{f}^{(a^{-1})} \quad (f^{(a)}(x) = f(a^{-1}x)) \tag{22}$$

which holds for all $a \in k^\times$ and μ-integrable f.

6.4.3 We define a *lattice* $L \subset k$ to be a cocompact discrete subgroup. We remark that lattices exist in k only if k is of positive characteristic. We remark also that the notion of lattice figures only in Corollary 6.11 below, and otherwise is unused in the sequel. For each lattice L put $L^\perp = \{\xi \in k | \mathbf{e}(x\xi) = 1 \text{ for all } x \in L\}$. For all lattices L again $L^\perp$ is a lattice and $(L^\perp)^\perp = L$. The *Poisson summation formula* states that

$$\sum_{x \in L} \Phi(x) = \mu(k/L)^{-1} \sum_{\xi \in L^\perp} \hat{\Phi}(\xi) \tag{23}$$

for all lattices $L \subset k$ and Schwartz functions $\Phi \in \mathrm{Sch}(k)$, where $\mu(k/L)$ denotes the *covolume* of L in k with respect to μ.

6.5 The linear functional $\mathcal{M}^{(a)}$ and linear operators $\mathcal{L}^\pm$

6.5.1 For each $\Phi \in \mathrm{Sch}(k)$ and $a \in k^\times$ put

$$\mathcal{M}^{(a)}[\Phi] = \int (\Phi(t) - \mathbf{1}_{a\mathcal{O}}(t)\Phi(0))d\mu^\times(t). \tag{24}$$

To see that $\mathcal{M}^{(a)}[\Phi]$ is a well-defined complex number, let $G(t)$ temporarily denote the integrand on the right side and note the following:

- $G(t)$ is defined and locally constant on $k^\times$.
- $G(t)$ vanishes for $\max(\|t\|, \|t\|^{-1})$ sufficiently large.

Therefore the integral on the right side of (24) converges. We have a scaling rule

$$\mathcal{M}^{(a)}[\Phi^{(b^{-1})}] = \mathcal{M}^{(ab)}[\Phi] = \mathcal{M}^{(a)}[\Phi] + \Phi(0)\,\mathrm{ord}\, b \tag{25}$$

for all $b \in k^\times$, cf. scaling rule (22) for the Fourier transform.

To motivate the definition of $\mathcal{M}^{(a)}$ we remark that Lemma 6.7 below can be reinterpreted as the assertion that the linear functional

$$\Phi \mapsto \Phi(0)\left(-\operatorname{ord}\kappa + \frac{1}{q-1}\right) + \mathcal{M}^{(1)}[\Phi]$$

on $\mathrm{Sch}(k)$ is the Fourier transform of $\operatorname{ord} x$ in the sense of the theory of distributions.

6.5.2 For each $\Phi \in \mathrm{Sch}(k)$ and $x \in k^\times$, put

$$\mathcal{L}^\pm[\Phi](x) = \int \frac{H(t^{\mp 1})\Phi(xt) - \frac{1}{2}\mathbf{1}_{\mathcal{O}^\times}(t)\Phi(x)}{\|t-1\|}\|t\|d\mu^\times(t), \tag{26}$$

where

$$H = \mathbf{1}_{\mathcal{O}} - \frac{1}{2}\mathbf{1}_{\mathcal{O}^\times}.$$

To see that $\mathcal{L}^\pm[\Phi](x)$ is a well-defined complex number, let $F^\pm(t)$ temporarily denote the integrand on the right side of (26), and note the following:

- $F^\pm(t)$ is defined and locally constant on $k^\times \setminus \{1\}$.
- $F^\pm(t) = 0$ for $\|t-1\|$ sufficiently small.
- $F^\pm(t) = 0$ for $\|t\|$ sufficiently large.
- $F^+(t) = 0$ for $\|t\|$ sufficiently small.
- $F^-(t) = \|t\|\Phi(0)$ for $\|t\|$ sufficiently small.

Therefore the integral on the right side of (26) converges. It is not difficult to verify that

$$\lim_{\|x\|\to 0} \mathcal{L}^-[\Phi](x) = \Phi(0)\int_{\|t\|<1} \|t\|d\mu^\times(t) = \frac{\Phi(0)}{q-1}. \tag{27}$$

We obtain another integral representation

$$\mathcal{L}^+[\Phi](x) = \int \frac{H(t)\Phi(xt^{-1}) - \frac{1}{2}\mathbf{1}_{\mathcal{O}^\times}(t)\Phi(x)}{\|1-t\|}d\mu^\times(t) \tag{28}$$

for $\mathcal{L}^+$ by substituting t^{-1} for t in the integral on the right side of (26).

In contrast to the case of the linear functional $\mathcal{M}^{(a)}$ we cannot easily give the motivation for the definition of the operators $\mathcal{L}^\pm$. The best we can say at present is that $\mathcal{L}^+$ makes the following theorem hold, and that $\mathcal{L}^-$ is indispensable to the proof of the theorem. A more conceptual characterization of the operators $\mathcal{L}^\pm$ would be nice to have.

Theorem 6.6 The local Stirling formula. *There exists a unique linear operator*

$$\mathcal{K} : \mathrm{Sch}(k) \to \mathrm{Sch}(k)$$

such that

$$\mathcal{K}[\Phi](x) = \begin{cases} -\Phi(x)\left(\frac{1}{2}\operatorname{ord}\kappa + \operatorname{ord} x\right) + \mathcal{L}^+[\Phi](x) & \text{if } x \neq 0, \\ -\frac{1}{2}\Phi(0)\operatorname{ord}\kappa + \mathcal{M}^{(1)}[\Phi] & \text{if } x = 0, \end{cases} \tag{29}$$

for all $\Phi \in \mathrm{Sch}(k)$ and $x \in k$. Moreover, we have

$$\mathcal{K}[\mathcal{F}[\Phi]] = -\mathcal{F}[\mathcal{K}[\Phi]] \tag{30}$$

for all $\Phi \in \mathrm{Sch}(k)$.

For the proof we need three lemmas.

Lemma 6.7. *We have*

$$\begin{array}{rl} & \int \Phi(x)\left(\frac{1}{2}\operatorname{ord}\kappa + \operatorname{ord} x\right) d\mu(x) \\ = & -\hat{\Phi}(0)\left(\frac{1}{2}\operatorname{ord}\kappa + \operatorname{ord} a\right) + \hat{\Phi}(0)/(q-1) + \mathcal{M}^{(a)}[\hat{\Phi}] \end{array} \tag{31}$$

for all $\Phi \in \mathrm{Sch}(k)$ and $a \in k^\times$.

Proof. By scaling rule (25) the right side of (31) is independent of a. For every $b \in k^\times$ we have

$$\hat{\Phi}(0)\operatorname{ord} b = \int (\|b\|^{-1}\Phi^{(b)}(x) - \Phi(x))\left(\frac{1}{2}\operatorname{ord}\kappa + \operatorname{ord} x\right) d\mu(x),$$

and hence by the scaling rules (22) and (25), the difference of left and right sides of (31) remains unchanged if we replace Φ by $\|b\|^{-1}\Phi^{(b)}$ (and hence $\hat{\Phi}$ by $\hat{\Phi}^{(b^{-1})}$). Clearly, the left side of (31) remains unchanged if we replace Φ by $\Phi^{(u)}$ for any $u \in \mathcal{O}^\times$. The same holds for the right side by the scaling rules (22) and (25). Consequently both sides of (31) remain unchanged if we replace Φ by the averaged function $\int_{\mathcal{O}^\times} \Phi^{(u)} d\mu^\times(u)$, which is a finite linear combination of functions of the form $\mathbf{1}_{b\mathcal{O}}$ with $b \in k^\times$. Taking into account all the preceding reductions, we may now assume without loss of generality that $a = 1$ and $\Phi = \mathbf{1}_{\mathcal{O}}$. Then equation (31) is easy to check by direct calculation. We omit further details. □

Lemma 6.8. *We have*

$$\begin{array}{rl} & \int \mathbf{e}(-x\xi)\Phi(x)\left(\frac{1}{2}\operatorname{ord}\kappa + \operatorname{ord} x\right) d\mu(x) \\ = & -\hat{\Phi}(\xi)\left(\frac{1}{2}\operatorname{ord}\kappa + \operatorname{ord}\xi\right) + (\mathcal{L}^+ + \mathcal{L}^-)[\hat{\Phi}](\xi) \end{array} \tag{32}$$

for all $\Phi \in \mathrm{Sch}(k)$ and $\xi \in k^\times$.

Proof. We calculate as follows:

$$\begin{array}{rl} & \hat{\Phi}(\xi)\left(\frac{1}{2}\operatorname{ord}\kappa + \operatorname{ord}\xi\right) + \int \mathbf{e}(-x\xi)\Phi(x)\left(\frac{1}{2}\operatorname{ord}\kappa + \operatorname{ord} x\right) d\mu(x) \\ = & \hat{\Phi}(\xi)/(q-1) + \int (\hat{\Phi}(\xi + t) - \hat{\Phi}(\xi)\mathbf{1}_{\mathcal{O}}(\xi^{-1}t))d\mu^\times(t) \\ = & \hat{\Phi}(\xi)/(q-1) + \int (\hat{\Phi}(\xi(t+1)) - \hat{\Phi}(\xi)\mathbf{1}_{\mathcal{O}}(t))d\mu^\times(t) \\ = & \hat{\Phi}(\xi)/(q-1) + \int \frac{\hat{\Phi}(\xi(t+1)) - \hat{\Phi}(\xi)\mathbf{1}_{\mathcal{O}}(t)}{\|t\|}\|t\|d\mu^\times(t) \\ = & \hat{\Phi}(\xi)/(q-1) + \int \frac{\hat{\Phi}(\xi t) - \hat{\Phi}(\xi)\mathbf{1}_{\mathcal{O}}(t)}{\|t-1\|}\|t\|d\mu^\times(t) \\ = & \int \frac{\hat{\Phi}(\xi t) - \hat{\Phi}(\xi)\mathbf{1}_{\mathcal{O}^\times}(t)}{\|t-1\|}\|t\|d\mu^\times(t) = (\mathcal{L}^+ + \mathcal{L}^-)[\hat{\Phi}](\xi) \end{array}$$

We get the first equality by applying (31) with a replaced by ξ and $\Phi(x)$ replaced by $\mathbf{e}(-x\xi)\Phi(x)$. The rest of the calculation is routine. □

Lemma 6.9. *Fix $\Phi \in \mathrm{Sch}(k)$. Fix $a \in k^\times$ such that Φ is supported in $a\mathcal{O}$. Fix $0 \neq b \in a\mathcal{O}$ such that Φ is constant on cosets of $b\mathcal{O}$. Fix $C > 0$ such that $|\Phi(x)| \leqslant C$ for all $x \in k$. Then we have*

$$\left|\frac{H(t)\Phi(xt^{-1}) - \frac{1}{2}\mathbf{1}_{\mathcal{O}^\times}(t)\Phi(x)}{\|1-t\|}\right| \leqslant C\|a/b\|\mathbf{1}_{ta\mathcal{O}}(x)\mathbf{1}_{\mathcal{O}}(t) \tag{33}$$

for all $x \in k^\times$ and $t \in k^\times \setminus \{1\}$.

For convenience in applying the lemma, note that

$$\int \mathbf{1}_{ta\mathcal{O}}(x)\mathbf{1}_{\mathcal{O}}(t)d\mu^\times(t) = \operatorname{ord}(x/a)\mathbf{1}_{\mathcal{O}}(x/a) \tag{34}$$

for all $a, x \in k^\times$.

Proof. By hypothesis we have

$$\|1-t\| < \|b/a\| \Rightarrow H(t)\Phi(xt^{-1}) - (1/2)\mathbf{1}_{\mathcal{O}^\times}(t)\Phi(x) = 0$$

and hence

$$\begin{aligned}&\left|\frac{H(t)\Phi(xt^{-1}) - (1/2)\mathbf{1}_{\mathcal{O}^\times}(t)\Phi(x)}{\|1-t\|}\right|\\ \leqslant\ & C\|a/b\|(H(t)\mathbf{1}_{ta\mathcal{O}}(x) + (1/2)\mathbf{1}_{\mathcal{O}^\times}(t)\mathbf{1}_{a\mathcal{O}}(x)),\end{aligned}$$

whence the result. □

6.10 Proof of the theorem

The right side of (29) defines on k a complex-valued function $\mathcal{K}[\Phi]$ depending linearly on Φ. By Lemma 6.9 the function $\mathcal{L}^+[\Phi]$ is μ-integrable and compactly supported, and hence so is the function $\mathcal{K}[\Phi]$. Similarly, $\mathcal{K}[\mathcal{F}[\Phi]]$ is μ-integrable and compactly supported. Since $\mathcal{K}[\Phi]$ is μ-integrable and compactly supported, it follows that the Fourier transform $\mathcal{F}[\mathcal{K}[\Phi]]$ is well-defined and locally constant. We have

$$\mathcal{F}[\mathcal{L}^+[\Phi]](\xi) = \mathcal{L}^-[\mathcal{F}[\Phi]](\xi) \quad (\xi \in k^\times), \tag{35}$$

since by another application of Lemma 6.9 it is justified to reverse the order of integration, and by limit formula (27) it follows that

$$\mathcal{F}[\mathcal{L}^+[\Phi]](0) = \frac{\hat{\Phi}(0)}{q-1}. \tag{36}$$

By (36) above, Lemma 6.7, and the definitions we have

$$\mathcal{F}[\mathcal{K}[\Phi]](0) = -\,\mathcal{K}[\mathcal{F}[\Phi]](0).$$

By (35) above, Lemma 6.8, and the definitions we have

$$\mathcal{F}[\mathcal{K}[\Phi]](\xi) = -\,\mathcal{K}[\mathcal{F}[\Phi]](\xi) \;\; (\xi \in k^\times).$$

Therefore we have $\mathcal{F}[\mathcal{K}[\Phi]] = -\,\mathcal{K}[\mathcal{F}[\Phi]]$, hence $\mathcal{K}[\mathcal{F}[\Phi]]$ is both compactly supported and locally constant, and hence $\mathcal{K}[\mathcal{F}[\Phi]] \in \mathrm{Sch}(k)$. Since $\Phi \in \mathrm{Sch}(k)$ was arbitrarily specified and the operator $\mathcal{F}$ is invertible, it follows that the operator $\mathcal{K}$ stabilizes the Schwartz space $\mathrm{Sch}(k)$ and anticommutes with the Fourier transform $\mathcal{F}$, as claimed. □

The following corollary will not be needed in the sequel—we include it just to motivate the theorem.

Corollary 6.11. *Fix a Schwartz function $\Phi \in \mathrm{Sch}(k)$ and a lattice $L \subset k$. Put*

$$\vartheta(t) := \sum_{x\in L}\Phi(t^{-1}x), \quad \vartheta^*(t) := \sum_{\xi\in L^\perp}\hat{\Phi}(t^{-1}\xi) \quad (t \in k^\times).$$

Then the following hold:

$$\vartheta(t) = \mu(k/L)^{-1}\|t\|\vartheta^*(t^{-1}) \tag{37}$$

$$\vartheta(t) = \Phi(0) \textit{ for } \|t\| \textit{ sufficiently small.} \tag{38}$$

$$\vartheta(t) = \mu(k/L)^{-1}\hat{\Phi}(0)\|t\| \textit{ for } \|t\| \textit{ sufficiently large.} \tag{39}$$

$$\vartheta(t) \textit{ is a locally constant function of } t. \tag{40}$$

We have

$$
\begin{aligned}
&\vartheta(a)\operatorname{ord}\kappa \\
&+\sum_{0\neq x\in L}\Phi(a^{-1}x)\operatorname{ord}x+\mu(k/L)^{-1}\sum_{0\neq\xi\in L^\perp}\|a\|\hat{\Phi}(a\xi)\operatorname{ord}\xi \\
&-\left(-\Phi(0)\operatorname{ord}a+\int(\Phi(t)-\Phi(0)\mathbf{1}_{\mathcal{O}}(t))d\mu^\times(t)\right) \\
&-\mu(k/L)^{-1}\|a\|\left(\hat{\Phi}(0)\operatorname{ord}a+\int\left(\hat{\Phi}(t)-\hat{\Phi}(0)\mathbf{1}_{\mathcal{O}}(t)\right)d\mu^\times(t)\right) \\
=&\int\frac{\vartheta(at)-\begin{cases}\Phi(0) & \text{if } \|t\|<1\\ \vartheta(a) & \text{if } \|t\|=1\\ \mu(k/L)^{-1}\hat{\Phi}(0)\|at\| & \text{if } \|t\|>1\end{cases}}{\|1-t\|}d\mu^\times(t)
\end{aligned}
\tag{41}
$$

for all $a\in k^\times$.

Proof of the corollary. Functional equation (37) follows from scaling formula (22) and the Poisson summation formula (23). Statements (38,39,40) are easy to verify. We omit the details. Statements (38,39,40) granted, it is clear that the integral on the right side of (41) converges. We turn to the proof of equation (41). First, we reduce to the case $a=1$ by observing that under replacement of the pair (Φ,a) by the pair $(\Phi^{(a)},1)$, neither the left side nor the right side of (41) change. Next, we note that

$$\sum_{x\in L}\mathcal{K}[\Phi](x)+\mu(k/L)^{-1}\sum_{\xi\in L^\perp}\mathcal{K}[\hat{\Phi}](\xi)=0$$

by the Poisson summation formula (23) and the anticommutation relation (30). Clearly, it is possible to rearrange the terms above to put the left side in coincidence with the left side of (41) in the case $a=1$, leaving a sum $A+\mu(k/L)^{-1}A^*$ on the right side, where

$$A=\sum_{0\neq x\in L}\mathcal{L}^+[\Phi](x),\quad A^*=\sum_{0\neq\xi\in L^\perp}\mathcal{L}^+[\hat{\Phi}](\xi).$$

Using the presentation (28) of the operator $\mathcal{L}^+$ and carrying the sum under the integral, we have

$$A=\int\frac{H(t)(\vartheta(t)-\Phi(0))-\frac{1}{2}\mathbf{1}_{\mathcal{O}^\times}(t)(\vartheta(1)-\Phi(0))}{\|1-t\|}d\mu^\times(t).$$

Exchange of sum and integral is justified by Lemma 6.9 and the remark (34) immediately following. We have a similar representation for A^*, which after the substitution of t^{-1} for t takes the form

$$A^*=\int\frac{H(t^{-1})(\vartheta^*(t^{-1})-\hat{\Phi}(0))-\frac{1}{2}\mathbf{1}_{\mathcal{O}^\times}(t)(\vartheta^*(1)-\hat{\Phi}(0))}{\|t-1\|}\|t\|d\mu^\times(t).$$

Finally, by exploiting functional equation (37) we can bring $A+\mu(k/L)^{-1}A^*$ to the form of the right side of (41) in the case $a=1$. We omit the remaining details of bookkeeping. □

6.12 Remark

Because the sum

$$\sum_{0\neq\xi\in L^\perp}\|a\|\hat{\Phi}(a\xi)\log\|\xi\|$$

vanishes for $\|a\| \gg 1$, equation (41) provides a precise asymptotic description of the sum

$$\sum_{0 \neq x \in L} \Phi(a^{-1}x) \log \|x\| \tag{42}$$

as $\|a\| \to \infty$. In the simple special case $\Phi = \mathbf{1}_{\mathcal{O}}$, the latter sum is analogous to the sum

$$\begin{aligned} &\frac{1}{2} \sum_{0 \neq x \in \mathbb{Z}} \mathbf{1}_{[-1,1]}(n^{-1}x) \log |x| \\ = \quad & \log n! = n \log n - n + \frac{1}{2} \log n + \frac{1}{2} \log(2\pi) + o_{n \to \infty}(1) \end{aligned} \tag{43}$$

on the real line. So it is reasonable to view equation (41) as an analogue and generalization of Stirling's formula in the nonarchimedean setting. Given the key role played by Theorem 6.6 in deriving (41), we choose to regard Theorem 6.6 itself as an analogue and generalization of the classical Stirling formula. This explains our terminology.

But the analogy of Theorem 6.6 with Stirling's formula is rather imperfect. A closer look reveals some complications. As it happens, for Theorem 6.6 and Corollary 6.11 we can derive direct analogues on the real line, using standard methods of the theory of tempered distributions, and these results in turn lead to a precise description of the asymptotic behavior as $n \to \infty$ of sums of the form

$$\sum_{0 \neq x \in \mathbb{Z}} \varphi(n^{-1}x) \log |x|,$$

where $\varphi(x)$ is a Schwartz function on the real line. For example, we can derive in this way the asymptotic formula

$$\begin{aligned} &\sum_{0 \neq x \in \mathbb{Z}} \exp(-\pi(x/n)^2) \log |x| \\ = \quad & n \log n - (\log 2 + \tfrac{1}{2} \log \pi + \gamma/2)n + \log 2\pi + o_{n \to \infty}(1), \end{aligned} \tag{44}$$

where γ is the Euler-Mascheroni constant. From the point of view of harmonic analysis, the sum (42) in the simple special case $\Phi = \mathbf{1}_{\mathcal{O}}$ is more closely analogous to the "soft-edged" sum (44) than to the "hard-edged" sum (43). These complications taken into account, we may only say that a "soft" analogy exists between Stirling's formula and Theorem 6.6. We shall discuss analogues of Theorem 6.6 for $\mathbb{R}$ and $\mathbb{C}$ and their applications in detail on another occasion.

7. The adelic Stirling formula

From now on in this paper we work in the setting of harmonic analysis on the adele ring of a global field of positive characteristic, a setting for which the (unfortunately) rather elaborate notation is set out in detail in tables immediately below. The main result of this section is the *adelic Stirling formula* (Theorem 7.7), which in form resembles Corollary 6.11. See [RV] for background on global harmonic analysis.

7.1 Data

- Let k be a global field of positive characteristic, of genus g, and with constant field $\mathbb{F}_q$.
- Fix a nonzero Kähler differential $\omega \in \Omega = \Omega_{k/\mathbb{F}_q}$.
- Fix a nonconstant character $\lambda : \mathbb{F}_q \to U(1)$, as in §3.

All constructions below proceed naturally from these choices.

7.2 Notation (local)

Let v be any place of k.

- Let k_v be the completion of k at v.
- Let $\mathcal{O}_v$ be the maximal compact subring of k_v.
- Let $\mathbb{F}_v$ be the residue field of $\mathcal{O}_v$.
- Let $\mathrm{Res}_v : \Omega \otimes_k k_v \to \mathbb{F}_v$ be the residue map at v.
- Put $\mathbf{e}_v = (x \mapsto \lambda(\mathrm{tr}_{\mathbb{F}_v/\mathbb{F}_q} \mathrm{Res}_v\, \omega x)) : k_v \to U(1)$.
- Let μ_v be Haar measure on k_v.
- Normalize μ_v by the rule $\mu_v \mathcal{O}_v \cdot \mu_v(\kappa_v^{-1}\mathcal{O}_v) = 1$.
- Let q_v be the cardinality of $\mathbb{F}_v$.
- For each $a \in k_v$, put $\|a\|_v = \frac{\mu_v(a\mathcal{O}_v)}{\mu_v \mathcal{O}_v}$ and $\mathrm{ord}_v\, a = -\frac{\log \|a\|_v}{\log q_v}$.
- Let $\mu_v^\times$ be Haar measure on $k_v^\times$.
- Normalize $\mu_v^\times$ by the rule $\mu_v^\times \mathcal{O}_v^\times = 1$.
- Let $\mathrm{ord}_v\, \omega$ be the order of vanishing of ω at v.
- Choose $\kappa_v \in k_v^\times$ such that $\mathrm{ord}_v\, \kappa_v = \mathrm{ord}_v\, \omega$, and hence $\kappa_v^{-1}\mathcal{O}_v = \{\xi \in k_v | \xi\mathcal{O}_v \subset \ker \mathbf{e}_v\}$.
- Let $\mathrm{Sch}(k_v)$ be the Schwartz class of functions on k_v.

7.3 Notation (global)

- Let $\mathbb{A}$ be the adele ring of k.
- Let $\mathcal{O} \subset \mathbb{A}$ be the maximal compact subring.
- Recall that each adele $x \in \mathbb{A}$ is a family $x = [x_v]$ indexed by places v such that $x_v \in k_v$ for all v and $x_v \in \mathcal{O}_v$ for all but finitely many v. Moreover, we have $\mathcal{O} = \prod_v \mathcal{O}_v$.
- As usual, we regard k as diagonally embedded in $\mathbb{A}$; thus k becomes a cocompact discrete subgroup of $\mathbb{A}$.
- Let μ be a Haar measure on $\mathbb{A}$.
- Normalize μ by the rule $\mu\mathcal{O} = q^{1-g}$. Equivalently: $\mu(\mathbb{A}/k) = 1$.
- Recall that each idele $a \in \mathbb{A}^\times$ is a family $a = [a_v]$ indexed by places v such that $a_v \in k_v^\times$ for all v and $a_v \in \mathcal{O}_v^\times$ for all but finitely many v. Moreover, we have $\mathcal{O}^\times = \prod_v \mathcal{O}_v^\times$.
- For each $a \in \mathbb{A}^\times$, put $\|a\| = \frac{\mu(a\mathcal{O})}{\mu\mathcal{O}}$.
- Let $\mu^\times$ be a Haar measure on $\mathbb{A}^\times$.
- Normalize $\mu^\times$ by the rule $\mu^\times \mathcal{O}^\times = 1$.
- Put $R_\omega = \big([x_v] \mapsto \sum_v \mathrm{tr}_{\mathbb{F}_v/\mathbb{F}_q} \mathrm{Res}_v\, x_v\omega\big) : \mathbb{A} \to \mathbb{F}_q$.
- For all $x, y \in \mathbb{A}$, put $\langle x, y\rangle = R_\omega(xy) \in \mathbb{F}_q$.
- Put $\mathbf{e} = \lambda \circ R_\omega = ([x_v] \mapsto \prod_v \mathbf{e}_v(x_v)) : \mathbb{A} \to U(1)$.
- Put $\kappa = [\kappa_v] \in \mathbb{A}^\times$.
- We write $\mathrm{ord}_v\, \omega = \mathrm{ord}_v\, \kappa = \mathrm{ord}_v\, \kappa_v$ for all places v.
- For each place v let
$$i_v : k_v \to \mathbb{A}, \quad i_v^\times : k_v^\times \to \mathbb{A}^\times$$
be the unique maps such that
$$(i_v(x))_w = \left\{ \begin{array}{ll} x & \text{if } w = v, \\ 0 & \text{otherwise,} \end{array}\right. \quad (i_v^\times(t))_w = \left\{ \begin{array}{ll} t & \text{if } w = v, \\ 1 & \text{otherwise,} \end{array}\right.$$
for all $x \in k_v$, $t \in k_v^\times$ and places w.

– Let $\mathrm{Sch}(\mathbb{A})$ be the Schwartz class of functions on $\mathbb{A}$.

7.4 Basic rules of calculation

To help reconcile the present system of notation to whatever system the reader might be familiar with, we recall a few facts routinely used below.

– $\int f(x)d\mu(x) = \|a\| \int f(ax)d\mu(x)$ all μ-integrable f and $a \in \mathbb{A}^\times$.
– $\mu = \bigotimes \mu_v$ and $\mu^\times = \bigotimes \mu_v^\times$.
– $\|a\| = \prod_v \|a_v\|_v$ for all $a = [a_v] \in \mathbb{A}^\times$.
– $\|x\| = 1$ for all $x \in k^\times$. (Artin's product formula.)
– $R_\omega x = 0$ for all $x \in k$. (Sum of residues equals zero.)
– $\|\kappa\|^{1/2} = q^{1-g} = \mu\mathcal{O}$.
– $\kappa^{-1}\mathcal{O} = \{\xi \in \mathbb{A} | \xi\mathcal{O} \subset \ker R_\omega\} = \{\xi \in \mathbb{A} | \xi\mathcal{O} \subset \ker \mathbf{e}\}$.
– For all $a, b \in \mathbb{A}^\times$ such that $a\mathcal{O} \subset b\mathcal{O}$, the spaces $b\mathcal{O}/a\mathcal{O}$ and $a^{-1}\kappa^{-1}\mathcal{O}/b^{-1}\kappa^{-1}\mathcal{O}$ are paired perfectly by $\langle \cdot, \cdot \rangle$. Moreover we have $k^\perp = k$ with respect to $\langle \cdot, \cdot \rangle$. (Serre duality.)

7.5 The adelic Fourier transform and the theta symbol

7.5.1 Given a μ-integrable complex-valued function f on $\mathbb{A}$, put

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \int f(x)\lambda(-\langle x, \xi \rangle)d\mu(x) = \int f(x)\mathbf{e}(-x\xi)d\mu(x)$$

for all $\xi \in \mathbb{A}$, thereby defining the *Fourier transform* $\hat{f}$, also denoted by $\mathcal{F}[f]$, which is a complex-valued continuous function on $\mathbb{A}$ tending to 0 at infinity. If f is compactly supported, then $\hat{f}$ is locally constant. If f is locally constant, then $\hat{f}$ is compactly supported. For example, we have

$$\mathcal{F}[\mathbf{1}_{x+b\mathcal{O}}](\xi) = q^{1-g}\|b\|\mathbf{1}_{b^{-1}\kappa^{-1}\mathcal{O}}(\xi)\lambda(-\langle x, \xi \rangle) \tag{45}$$

for all $b \in \mathbb{A}^\times$ and $x, \xi \in \mathbb{A}$.

7.5.2 The Schwartz class $\mathrm{Sch}(\mathbb{A})$ is stable under the Fourier transform and the *Fourier inversion formula* states that

$$\Phi(x) = \int \hat{\Phi}(\xi)\mathbf{e}(x\xi)d\mu(\xi) \tag{46}$$

for all $\Phi \in \mathrm{Sch}(\mathbb{A})$. Our normalization of μ is chosen to make the Fourier inversion formula hold in the simple form stated above; in general there would be a positive constant depending on μ but independent of Φ multiplying the right side. The Fourier inversion formula implies the *squaring rule*

$$\mathcal{F}^2[\Phi](x) = \Phi(-x) \tag{47}$$

for all $\Phi \in \mathrm{Sch}(\mathbb{A})$. We mention also the *scaling rule*

$$\mathcal{F}[f^{(a)}] = \|a\|\hat{f}^{(a^{-1})} \quad (f^{(a)}(x) = f(a^{-1}x)) \tag{48}$$

holding for all $a \in \mathbb{A}^\times$ and μ-integrable f.

7.5.3 The *Poisson summation formula* states that

$$\sum_{x \in k} \Phi(a^{-1}x) = \|a\| \sum_{\xi \in k} \hat{\Phi}(a\xi)$$

for all $a \in \mathbb{A}^\times$ and $\Phi \in \mathrm{Sch}(\mathbb{A})$. Note that since both Φ and $\hat{\Phi}$ are compactly supported, only finitely many nonzero terms occur in the sums on either side of the formula. The normalization

$\mu\mathcal{O} = q^{1-g} = \mu(\mathbb{A}/k)$ of Haar measure μ on $\mathbb{A}$ was chosen to make the Poisson summation formula hold in the particularly simple form above; in general the right side would be multiplied by a positive constant depending on μ but independent of Φ.

7.5.4 For all $a \in \mathbb{A}^\times$ and $\Phi \in \mathrm{Sch}(\mathbb{A})$ put

$$\Theta(a, \Phi) = \sum_{x \in k} \Phi(a^{-1}x),$$

thereby defining the *theta symbol* $\Theta(\cdot, \cdot)$. We have

$$\Theta(ax, \Phi) = \Theta(a, \Phi) \tag{49}$$

for all $x \in k^\times$. In other words, $\Theta(a, \Phi)$ depends only on the image of a in the *idele class group* $\mathbb{A}^\times/k^\times$. Clearly, we have a *scaling rule*

$$\Theta(a, \Phi^{(b)}) = \Theta(ab, \Phi) = \Theta(1, \Phi^{(ab)}) \tag{50}$$

for all $b \in \mathbb{A}^\times$. We have a *functional equation*

$$\Theta(a, \Phi) = \|a\|\Theta(a^{-1}, \hat{\Phi}), \tag{51}$$

which is just a rewrite of the Poisson summation formula.

Proposition 7.6. *Fix $\Phi \in \mathrm{Sch}(\mathbb{A})$. Fix $b \in \mathbb{A}^\times$ and $f \in \mathbb{A}^\times \cap \mathcal{O}$ such that Φ is constant on cosets of $b\mathcal{O}$ and supported in $f^{-1}b\mathcal{O}$. Then: (i) We have*

$$\Theta(a, \Phi) = \begin{cases} \|a\|\hat{\Phi}(0) & \text{if } \|ab\| > q^{2g-2}, \\ \\ \Phi(0) & \text{if } \|ab\| < \|f\|, \end{cases}$$

for all $a \in \mathbb{A}^\times$. (ii) Moreover, $\Theta(a, \Phi)$ depends in locally constant fashion on a.

Proof. From example (45) and Serre duality it follows that the Fourier transform $\hat{\Phi}$ is constant on cosets of $fb^{-1}\kappa^{-1}\mathcal{O}$ and supported in $b^{-1}\kappa^{-1}\mathcal{O}$. We therefore have

$$\sum_{x \in k \cap af^{-1}b\mathcal{O}} \Phi(a^{-1}x) = \Theta(a, \Phi) = \|a\| \sum_{\xi \in k \cap a^{-1}b^{-1}\kappa^{-1}\mathcal{O}} \hat{\Phi}(a\xi). \tag{52}$$

via (51). Statement (i) follows via the Artin product formula. Statement (ii) follows from the observation that the map $a \mapsto k \cap af^{-1}b\mathcal{O}$ from $\mathbb{A}^\times$ to finite subsets of k is locally constant. $\square$

Theorem 7.7 The adelic Stirling formula. *For all ideles $a \in \mathbb{A}^\times$, Schwartz functions $\Phi \in \mathrm{Sch}(\mathbb{A})$ and places v of k we have*

$$\Theta(a, \Phi) \operatorname{ord}_v \omega + \sum_{x \in k^\times} \Phi(a^{-1}x) \operatorname{ord}_v x + \sum_{\xi \in k^\times} \|a\|\hat{\Phi}(a\xi) \operatorname{ord}_v \xi$$

$$= \int \frac{\Theta(i_v^\times(t)a, \Phi) - \begin{cases} \Phi(0) & \text{if } \|t\|_v < 1 \\ \Theta(a, \Phi) & \text{if } \|t\|_v = 1 \\ \|a\|\|t\|_v\hat{\Phi}(0) & \text{if } \|t\|_v > 1 \end{cases}}{\|1-t\|_v} d\mu_v^\times(t) \tag{53}$$

$$-\Phi(0) \operatorname{ord}_v a_v + \int (\Phi(i_v(t)) - \Phi(0)\mathbf{1}_{\mathcal{O}_v}(t))\, d\mu_v^\times(t)$$

$$+\|a\| \left(\hat{\Phi}(0) \operatorname{ord}_v a_v + \int (\hat{\Phi}(i_v(t)) - \hat{\Phi}(0)\mathbf{1}_{\mathcal{O}_v}(t)) d\mu_v^\times(t) \right).$$

Note that Proposition 7.6 guarantees convergence of the integral involving the theta symbol on the right side of (53). The pattern set by the proof of Corollary 6.11 will guide us in proving (53). The proof will be completed in §7.11 after some preparation. The rationale for the Stirling formula terminology here is essentially the same as that offered in connection with Theorem 6.6 and Corollary 6.11. Namely, to the extent that sums of the form $\sum_{x\in k^\times}\Phi(a^{-1}x)\operatorname{ord}_v x$ may be regarded as analogues of $\log n!$, formula (53) may be regarded as an analogue of the classical Stirling formula. It is an interesting problem to devise and interpret a version of the adelic Stirling formula for number fields. We hope to have progress to report on this problem in the near future.

7.8 Recollection of the tensor decomposition of $\mathrm{Sch}(\mathbb{A})$

7.8.1 Given any family of vector spaces $\{V_i\}_{i\in I}$, where each V_i is equipped with a *neutral element* $0\neq e_i\in V_i$, there is a natural way to form the (possibly infinite) tensor product $\bigotimes_{i\in I}V_i$. Namely, the latter is by definition spanned by symbols of the form

$$\bigotimes_{i\in I} v_i \quad (v_i\in V_i \text{ for all } i, \text{ and } v_i=e_i \text{ for all but finitely many } i),$$

subject to the obvious relations. Moreover, given for each index i a linear endomorphism L_i of V_i such that $L_ie_i=e_i$ for all but finitely many i, there is a natural tensor product $\bigotimes_{i\in I}L_i$ of operators, namely, that sending each symbol $\bigotimes_{i\in I}v_i$ to $\bigotimes_{i\in I}(L_iv_i)$.

7.8.2 In the general sense above, as is well known, $\mathrm{Sch}(\mathbb{A})$ can naturally be identified with $\bigotimes_v \mathrm{Sch}(k_v)$, where for each place v the space $\mathrm{Sch}(k_v)$ is equipped with the neutral element $\mathbf{1}_{\mathcal{O}_v}$. (See, for example, [RV, p. 260]. But note that in the definition presented there the condition "$f_v|_{\mathcal{O}_v}=1$" should be strengthened to "$f_v=\mathbf{1}_{\mathcal{O}_v}$". Otherwise the function f thus defined might fail to be compactly supported.)

7.8.3 We have at least one natural example of a linear operator on $\mathrm{Sch}(\mathbb{A})$ that factors as an infinite tensor product, namely the Fourier transform $\mathcal{F}$. More precisely, for each place v, let

$$\mathcal{F}_v:\mathrm{Sch}(k_v)\to\mathrm{Sch}(k_v)$$

be the local Fourier transform defined by the rule

$$\mathcal{F}_v[\Phi](\xi)=\int\Phi(x)\mathbf{e}_v(-x\xi)d\mu_v(x).$$

Since $\mathcal{F}_v[\mathbf{1}_{\mathcal{O}_v}]=\mathbf{1}_{\mathcal{O}_v}$ for all but finitely many v, it makes sense to form the tensor product of the operators $\mathcal{F}_v$ over all places v. One can easily verify that this tensor product does indeed coincide with $\mathcal{F}$.

7.8.4 Let a place v and a linear endomorphism of L of $\mathrm{Sch}(k_v)$ be given. Let $\mathbb{A}^v$ be the "coordinate hyperplane" of $\mathbb{A}$ consisting of families $x=[x_w]$ such that $x_v=0$. Given $y\in\mathbb{A}^v$ and $\Phi\in\mathrm{Sch}(\mathbb{A})$, let $\Phi_{y,v}\in\mathrm{Sch}(k_v)$ be defined by the rule

$$\Phi_{y,v}(x)=\Phi(i_v(x)+y)$$

for all $x\in k_v$. Then, so we claim, there exists a unique linear endomorphism $\tilde{L}$ of $\mathrm{Sch}(\mathbb{A})$ such that

$$\tilde{L}[\Phi](i_v(x)+y)=L[\Phi_{y,v}](x)$$

for all $\Phi\in\mathrm{Sch}(\mathbb{A})$, $x\in k_v$ and $y\in\mathbb{A}^v$. To prove the claim one just has to check that $\tilde{L}$ is the tensor product of the family

$$L_w=\begin{cases} L & \text{if } v=w,\\ 1 & \text{if } v\neq w,\end{cases}$$

extended over all places w. We omit these details. We call $\tilde{L}$ the *canonical prolongation* of L.

7.9 The operators $\mathcal{M}_v$, $\mathcal{L}_v$ and $\mathcal{K}_v$

Fix a place v of k. We keep the notation introduced in the preceding discussion of the tensor decomposition of $\mathrm{Sch}(\mathbb{A})$.

7.9.1 Given $\Phi \in \mathrm{Sch}(\mathbb{A})$, we define a function $\mathcal{M}_v[\Phi]$ on $\mathbb{A}^v$ by the rule

$$\mathcal{M}_v[\Phi](y) = \int (\Phi_{y,v}(t) - \mathbf{1}_{\mathcal{O}_v}(t)\Phi_{y,v}(0))d\mu_v^\times(t)$$

for all $y \in \mathbb{A}^v$. The function $\mathcal{M}_v[\Phi]$ is well-defined by a repetition of the argument given in §6.5.1. Note in particular that

$$\mathcal{M}_v[\Phi](0) = \int (\Phi(i_v(t)) - \mathbf{1}_{\mathcal{O}_v}(t)\Phi(0))d\mu_v^\times(t) \tag{54}$$

for all $\Phi \in \mathrm{Sch}(\mathbb{A})$.

7.9.2 Given $\Phi \in \mathrm{Sch}(\mathbb{A})$, we define a function $\mathcal{L}_v[\Phi]$ on $\mathbb{A} \setminus \mathbb{A}^v$ by the rule

$$\mathcal{L}_v[\Phi](i_v(x) + y) = \int \frac{H_v(t)\Phi_{y,v}(t^{-1}x) - \frac{1}{2}\mathbf{1}_{\mathcal{O}_v^\times}(t)\Phi_{y,v}(x)}{\|1-t\|_v} d\mu_v^\times(t)$$

for all $x \in k_v^\times$ and $y \in \mathbb{A}^v$, where

$$H_v = \mathbf{1}_{\mathcal{O}_v} - \frac{1}{2}\mathbf{1}_{\mathcal{O}_v^\times}.$$

The function $\mathcal{L}_v[\Phi]$ is well-defined by a repetition of the argument given in §6.5.2. Equivalently, we have

$$\mathcal{L}_v[\Phi](z) = \int \frac{H_v(t)\Phi(i_v^\times(t^{-1})z) - \frac{1}{2}\mathbf{1}_{\mathcal{O}_v^\times}(t)\Phi(z)}{\|1-t\|_v} d\mu_v^\times(t) \tag{55}$$

for all $z \in \mathbb{A} \setminus \mathbb{A}^v$. Note that the operator $\mathcal{L}_v$ preserves supports in the sense that for all $a \in \mathbb{A}^\times$ and $\Phi \in \mathrm{Sch}(\mathbb{A})$, if Φ vanishes outside $a\mathcal{O}$, then $\mathcal{L}_v[\Phi]$ vanishes outside $a\mathcal{O} \setminus \mathbb{A}^v$.

7.9.3 Theorem 6.6 and the canonical prolongation process now yield a unique linear operator

$$\mathcal{K}_v : \mathrm{Sch}(\mathbb{A}) \to \mathrm{Sch}(\mathbb{A})$$

such that

$$\begin{aligned} &\mathcal{K}_v[\Phi](i_v(x) + y) \\ &= \begin{cases} -\Phi(i_v(x)+y)(\frac{1}{2}\operatorname{ord}_v \omega + \operatorname{ord}_v x) + \mathcal{L}_v[\Phi](i_v(x)+y) & \text{if } x \neq 0, \\ -\frac{1}{2}\Phi(y)\operatorname{ord}_v \omega + \mathcal{M}_v[\Phi](y) & \text{if } x = 0, \end{cases} \end{aligned}$$

for all $\Phi \in \mathrm{Sch}(\mathbb{A})$, $x \in k_v$ and $y \in \mathbb{A}^v$. Moreover, since $\mathcal{F}$ factors as the tensor product of local Fourier transforms, and the operator on $\mathrm{Sch}(k_v)$ defined by Theorem 6.6 anticommutes with $\mathcal{F}_v$, we necessarily have

$$\mathcal{K}_v[\mathcal{F}[\Phi]] = -\mathcal{F}[\mathcal{K}_v[\Phi]]$$

for all $\Phi \in \mathrm{Sch}(\mathbb{A})$, i. e., $\mathcal{K}_v$ anticommutes with the adelic Fourier transform $\mathcal{F}$.

Lemma 7.10. *Fix $\Phi \in \mathrm{Sch}(\mathbb{A})$. Fix a place v of k. Put*

$$\vartheta(t) = \Theta(i_v^\times(t), \Phi)$$

for all $t \in k_v^\times$. *Then we have*

$$\sum_{x\in k^\times}\mathcal{L}_v[\Phi](x)+\sum_{\xi\in k^\times}\mathcal{L}_v[\hat{\Phi}](\xi)$$

$$=\int\frac{\vartheta(t)-\left\{\begin{array}{cc}\Phi(0) & \text{if } \|t\|_v<1\\ \vartheta(1) & \text{if } \|t\|_v=1\\ \|t\|_v\hat{\Phi}(0) & \text{if } \|t\|_v>1\end{array}\right.}{\|1-t\|_v}d\mu_v^\times(t). \tag{56}$$

Convergence of the sums on the left and the integral on the right will be established in the course of the proof.

Proof. The integrand on the right side of (56) is locally constant and is supported on a compact subset of $k_v^\times\setminus\{1\}$ by Proposition 7.6. So the integral on the right side of (56) converges. Now fix $a\in\mathbb{A}^\times$ such that both Φ and $\hat{\Phi}$ are supported in $a\mathcal{O}$, hence both $\mathcal{L}_v[\Phi]$ and $\mathcal{L}_v[\hat{\Phi}]$ are supported in $a\mathcal{O}\setminus\mathbb{A}^v$, and hence the quantities

$$\Phi(i_v(t)^{-1}x),\quad \hat{\Phi}(i_v(t)^{-1}\xi),\quad \mathcal{L}_v[\Phi](x),\quad \mathcal{L}_v[\hat{\Phi}](\xi)$$

vanish for all $x,\xi\in k\setminus a\mathcal{O}$ and $t\in k_v^\times\cap\mathcal{O}_v$. Convergence of the sums on the left side of (56) follows, and moreover it is justified to carry the summations under the integral (55) representing the operator $\mathcal{L}_v$. We find that the left side of (56) equals

$$\int\frac{H_v(t)(\vartheta(t)-\Phi(0))-\frac{1}{2}\mathbf{1}_{\mathcal{O}_v^\times}(t)(\vartheta(1)-\Phi(0))}{\|1-t\|_v}d\mu_v^\times(t)$$

$$+\int\frac{H_v(t)(\|t\|_v\vartheta(t^{-1})-\hat{\Phi}(0))-\frac{1}{2}\mathbf{1}_{\mathcal{O}_v^\times}(t)(\vartheta(1)-\hat{\Phi}(0))}{\|1-t\|_v}d\mu_v^\times(t),$$

whence the result after making a substitution of t^{-1} for t in the second integral and then collecting like terms. ☐

7.11 Proof of the theorem

Neither side of (53) changes under the replacement of (a,Φ) by $(1,\Phi^{(a)})$. We may therefore assume without loss of generality that $a=1$. Since $\mathcal{K}_v$ anti-commutes with $\mathcal{F}$, the Poisson summation formula gives the identity

$$\sum_{x\in k}\mathcal{K}_v[\Phi](x)+\sum_{\xi\in k}\mathcal{K}_v[\hat{\Phi}](\xi)=0.$$

After rearranging the terms in this last identity and applying (51), (54) and (56), we obtain (53) in the case $a=1$. The proof of the theorem is complete. ☐

8. The rational Fourier transform and the Catalan symbol (adelic versions)

We pick up again the ideas introduced in §3.

8.1 The rational Fourier transform

8.1.1 With $\lambda_0:\mathbb{F}_q\to\{-1,0,1\}\subset\mathbb{C}$ as in §3.3, put

$$\mathbf{e}_0=\lambda_0\circ R_\omega:\mathbb{A}\to\{-1,0,1\}\subset\mathbb{C}$$

and then given any $\Phi \in \mathrm{Sch}(\mathbb{A})$, put

$$\mathcal{F}_0[\Phi](\xi) = \tilde{\Phi}(\xi) = \int \Phi(x)\mathbf{e}_0(x\xi)d\mu(x) = \int \Phi(x)\lambda_0(\langle x, \xi\rangle)d\mu(x)$$

for all $\xi \in \mathbb{A}$, thereby defining the *rational Fourier transform* $\tilde{\Phi}$ of Φ, also denoted by $\mathcal{F}_0[\Phi]$. Via the trivial identities (9) we have

$$\begin{array}{rcl} \hat{\Phi}(\xi) & = & -\sum_{c\in\mathbb{F}_q^\times} \lambda(-c)\tilde{\Phi}(c^{-1}\xi), \\ \tilde{\Phi}(\xi) & = & q^{-1}\sum_{c\in\mathbb{F}_q^\times}(1-\lambda(c))\hat{\Phi}(c\xi) \end{array} \tag{57}$$

for all $\xi \in \mathbb{A}$. Thus $\hat{\Phi}$ and $\tilde{\Phi}$ are expressible each in terms of the other. It follows that $\mathcal{F}_0$ preserves the Schwartz space $\mathrm{Sch}(\mathbb{A})$, just as does the standard Fourier transform $\mathcal{F}$. It follows from the trivial identities (9) and the example (45) that

$$\mathcal{F}_0[\mathbf{1}_{x+b\mathcal{O}}] = q^{1-g}\|b\|\left(\mathbf{1}_{b^{-1}\kappa^{-1}\mathcal{O}\cap[\langle x,\cdot\rangle=0]} - \mathbf{1}_{b^{-1}\kappa^{-1}\mathcal{O}\cap[\langle x,\cdot\rangle=1]}\right) \tag{58}$$

for all $b \in \mathbb{A}^\times$ and $x, \xi \in \mathbb{A}$. It follows in turn via Serre duality that for all $b \in \mathbb{A}^\times$, $f \in \mathbb{A}^\times \cap \mathcal{O}$ and $\Phi \in \mathrm{Sch}(\mathbb{A})$ such that Φ is constant on cosets of $b\mathcal{O}$ and supported in $f^{-1}b\mathcal{O}$, the rational Fourier transform $\tilde{\Phi}$ is constant on cosets of $fb^{-1}\kappa^{-1}\mathcal{O}$ and supported in $b^{-1}\kappa^{-1}\mathcal{O}$.

8.1.2 Fix $\Phi \in \mathrm{Sch}(\mathbb{A})$. By equations (57) relating standard and rational Fourier transforms, the inversion formula (46) for the standard Fourier transform, and the trivial identities (9), we have

$$\begin{array}{rcl} \Phi(x) & = & \int \mathbf{e}(x\xi)\left(-\sum_{c\in\mathbb{F}_q^\times}\lambda(-c)\tilde{\Phi}(c^{-1}\xi)\right)d\mu(\xi) \\ & = & \int\left(-\sum_{c\in\mathbb{F}_q^\times}\lambda(-c)\mathbf{e}(cx\xi)\right)\tilde{\Phi}(\xi)d\mu(\xi) \\ & = & \int\left(q\mathbf{e}_0(x\xi) - \sum_{c\in\mathbb{F}_q^\times}\mathbf{e}_0(cx\xi)\right)\tilde{\Phi}(\xi)d\mu(\xi) \end{array}$$

for all $x \in \mathbb{A}$; thus we can invert the rational Fourier transform.

8.1.3 Given $\Phi \in \mathrm{Sch}(\mathbb{A})$ and $x \in \mathbb{A}$ we put

$$\mathcal{N}[\Phi](x) = \sum_{c\in\mathbb{F}_q^\times}\Phi(cx),$$

thereby defining

$$\mathcal{N}[\Phi] \in \mathrm{Sch}(\mathbb{A}).$$

Then the inversion formula above for the rational Fourier transform can be rewritten more compactly in the form

$$(q-\mathcal{N})\,\mathcal{F}_0^2 = 1.$$

The related operator identities

$$\mathcal{N}\,\mathcal{F}_0 = \mathcal{F}_0\,\mathcal{N} = \mathcal{F}\,\mathcal{N} = \mathcal{N}\,\mathcal{F}, \quad q = (q-\mathcal{N})(1+\mathcal{N})$$

and the *squaring rule*

$$\mathcal{F}_0^2 = q^{-1}(1+\mathcal{N}) \tag{59}$$

are easy to verify. We omit the proofs. The last formula should be compared with the squaring rule (47) for the adelic Fourier transform. (If $q = 2$, then the two formulas are the same.)

8.1.4 We have for the rational Fourier transform a *scaling rule* of exactly the same form as that obeyed by the usual Fourier transform, namely

$$\mathcal{F}_0[\Phi^{(a)}] = \|a\|\tilde{\Phi}^{(a^{-1})} \tag{60}$$

for all $\Phi \in \mathrm{Sch}(\mathbb{A})$ and $a \in \mathbb{A}^\times$. Moreover, since $\mathcal{N}\mathcal{F} = \mathcal{N}\mathcal{F}_0$ we have

$$\sum_{x\in k} \Phi(a^{-1}x) = \sum_{x\in k} \Phi^{(a)}(x) = \sum_{\xi\in k} \mathcal{F}_0[\Phi^{(a)}](\xi) = \|a\| \sum_{\xi\in k} \tilde{\Phi}(a\xi)$$

for all $a \in \mathbb{A}^\times$ and $\Phi \in \mathrm{Sch}(\mathbb{A})$, i. e., the Poisson summation formula continues to hold with the rational Fourier transform replacing the standard one. Consequently we have a *functional equation*

$$\Theta(a, \Phi) = \|a\|\Theta(a^{-1}, \tilde{\Phi}) \tag{61}$$

for all $a \in \mathbb{A}^\times$ and $\Phi \in \mathrm{Sch}(\mathbb{A})$.

8.2 The Catalan symbol

8.2.1 Recall that $\mathbb{Z}[1/q]$ (as in the discussion of toy rational Fourier transforms and toy Catalan symbols) is the ring consisting of those rational numbers x such that $q^n x \in \mathbb{Z}$ for $n \gg 0$. Let $\mathrm{Sch}_0(\mathbb{A})$ denote the space of $\mathbb{Z}[1/q]$-valued Schwartz functions on $\mathbb{A}$, and let $\mathrm{Sch}_{00}(\mathbb{A})$ denote the subgroup consisting of functions Φ such that $\Phi(0) = 0 = \tilde{\Phi}(0)$. Clearly, the groups $\mathrm{Sch}_0(\mathbb{A}) \supset \mathrm{Sch}_{00}(\mathbb{A})$ are stable under the action of the operators $\mathcal{F}_0$ and $\mathcal{N}$. Note that $\Theta(a, \Phi) \in \mathbb{Z}[1/q]$ for all $a \in \mathbb{A}^\times$ and $\Phi \in \mathrm{Sch}_0(\mathbb{A})$. Note that x^n makes sense for all $x \in k_{\mathrm{perf}}^\times$ and $n \in \mathbb{Z}[1/q]$. Note that $\|a\| \in q^{\mathbb{Z}}$ for all $a \in \mathbb{A}^\times$ and hence $x^{\|a\|}$ makes sense for all $x \in k_{\mathrm{perf}}$. For all places v of k and $x \in k_{\mathrm{perf}}^\times$ we define $\mathrm{ord}_v\, x = q^{-n}\,\mathrm{ord}_v\, x^{q^n} \in \mathbb{Z}[1/q]$ for all sufficiently large n.

8.2.2 For all $\Phi \in \mathrm{Sch}_0(\mathbb{A})$ and $a \in \mathbb{A}^\times$, put

$$\begin{aligned} \left(\begin{array}{c} a \\ \Phi \end{array}\right)_+ &= \prod_{x\in k^\times} x^{\Phi(a^{-1}x)} \in k_{\mathrm{perf}}^\times, \\ \left(\begin{array}{c} a \\ \Phi \end{array}\right) &= \left(\begin{array}{c} a \\ \Phi \end{array}\right)_+ \left(\begin{array}{c} a^{-1} \\ \tilde{\Phi} \end{array}\right)_+^{\|a\|} \in k_{\mathrm{perf}}^\times. \end{aligned}$$

In the infinite products only finitely many terms differing from 1 occur, so these objects are well-defined. We call $(\dot{\cdot})$ the *Catalan symbol*, and $(\dot{\cdot})_+$ the *partial Catalan symbol*. The rationale for the terminology is the same as the one offered in the case of the toy Catalan symbol, namely the values of the Catalan and partial Catalan symbol bear a certain structural resemblance to the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$.

Proposition 8.3. *Fix $\Phi \in \mathrm{Sch}_0(\mathbb{A})$. Fix $b \in \mathbb{A}^\times$ and $f \in \mathbb{A}^\times \cap \mathcal{O}$ such that Φ is constant on cosets of $b\mathcal{O}$ and supported in $f^{-1}b\mathcal{O}$. Then*

$$\left(\begin{array}{c} a \\ \Phi \end{array}\right) = \begin{cases} \left(\begin{array}{c} a^{-1} \\ \tilde{\Phi} \end{array}\right)_+^{\|a\|} & \text{if } \|ab\| < \|f\| \\ \left(\begin{array}{c} a \\ \Phi \end{array}\right)_+ & \text{if } \|ab\| > q^{2g-2} \end{cases} \tag{62}$$

for all $a \in \mathbb{A}^\times$.

Proof. In parallel to formula (52), we have

$$\left(\begin{array}{c} a \\ \Phi \end{array}\right)_+ = \prod_{x\in k^\times\cap af^{-1}b\mathcal{O}} x^{\Phi(a^{-1}x)}, \quad \left(\begin{array}{c} a^{-1} \\ \tilde{\Phi} \end{array}\right)_+ = \prod_{\xi\in k^\times\cap a^{-1}\kappa^{-1}b^{-1}\mathcal{O}} \xi^{\tilde{\Phi}(a\xi)},$$

whence the result by Artin's product formula and the definitions. ∎

8.4 Formal properties of the Catalan symbol

8.4.1 Given $a, b \in \mathbb{A}^\times$, $x \in k^\times$, and $\Phi \in \mathrm{Sch}_0(\mathbb{A})$, we claim that

$$\begin{pmatrix} ax \\ \Phi \end{pmatrix} = \begin{pmatrix} a \\ \Phi \end{pmatrix} x^{-\Phi(0)+\|a\|\tilde{\Phi}(0)}, \tag{63}$$

$$\begin{pmatrix} ab \\ \Phi \end{pmatrix} = \begin{pmatrix} a \\ \Phi^{(b)} \end{pmatrix} = \begin{pmatrix} 1 \\ \Phi^{(ab)} \end{pmatrix}, \tag{64}$$

and

$$\begin{pmatrix} a^{-1} \\ \tilde{\Phi} \end{pmatrix}^{\|a\|} = (-1)^{\Theta(a,\Phi)-\Phi(0)} \begin{pmatrix} a \\ \Phi \end{pmatrix}. \tag{65}$$

Note in particular that if $\Phi \in \mathrm{Sch}_{00}(\mathbb{A})$, then $\begin{pmatrix} a \\ \Phi \end{pmatrix}$ depends only on the image of a in the idele class group $\mathbb{A}^\times/k^\times$. The last two relations we call the *scaling rule* and *functional equation*, respectively, obeyed by the Catalan symbol.

8.4.2 Here are the proofs of the claims. The calculation

$$\begin{aligned}
\begin{pmatrix} ax \\ \Phi \end{pmatrix} &= \textstyle\prod_{y \in k^\times} y^{\Phi(a^{-1}x^{-1}y)} \cdot \prod_{\eta \in k^\times} \eta^{\tilde{\Phi}(ax\eta)\|ax\|} \\
&= \textstyle\prod_{y \in k^\times} (xy)^{\Phi(a^{-1}y)} \cdot \prod_{\eta \in k^\times} (x^{-1}\eta)^{\tilde{\Phi}(a\eta)\|ax\|} \\
&= \begin{pmatrix} a \\ \Phi \end{pmatrix} \cdot x^{\Theta(a,\Phi)-\Phi(0)-\|ax\|\Theta(a^{-1},\tilde{\Phi})+\|ax\|\tilde{\Phi}(0)} \\
&= \begin{pmatrix} a \\ \Phi \end{pmatrix} x^{-\Phi(0)+\|a\|\tilde{\Phi}(0)}
\end{aligned}$$

proves (63). At the last equality we applied Artin's product formula and functional equation (61). The calculation

$$\begin{aligned}
\begin{pmatrix} ab \\ \Phi \end{pmatrix} &= \prod_{x \in k^\times} x^{\Phi(a^{-1}b^{-1}x)} \cdot \prod_{\xi \in k^\times} \xi^{\tilde{\Phi}(ab\xi)\|ab\|} \\
&= \prod_{x \in k^\times} x^{\Phi^{(b)}(a^{-1}x)} \cdot \prod_{\xi \in k^\times} \xi^{\mathcal{F}_0[\Phi^{(b)}](a\xi)\|a\|} = \begin{pmatrix} a \\ \Phi^{(b)} \end{pmatrix}
\end{aligned}$$

proves (64). At the second equality we applied scaling rule (60). Finally, the calculation

$$\begin{aligned}
\begin{pmatrix} a^{-1} \\ \tilde{\Phi} \end{pmatrix}^{q\|a\|} &= \prod_{x\in k^\times} x^{\tilde{\Phi}(ax)\|a\|q} \cdot \prod_{\xi\in k^\times} \xi^{q\,\mathcal{F}_0^2[\Phi](a^{-1}\xi)} \\
&= \prod_{x\in k^\times} x^{\tilde{\Phi}(ax)\|a\|q} \cdot \prod_{\xi\in k^\times} \xi^{(1+\mathcal{N})[\Phi](a^{-1}\xi)} \\
&= \begin{pmatrix} a \\ \Phi \end{pmatrix}^{q} \prod_{c\in\mathbb{F}_q^\times} \prod_{\xi\in k^\times} c^{\Phi(a^{-1}\xi)} \\
&= \begin{pmatrix} a \\ \Phi \end{pmatrix}^{q} (-1)^{\Theta(a,\Phi)-\Phi(0)}
\end{aligned}$$

proves (65). At the second equality we applied formula (59) for $\mathcal{F}_0^2$.

8.4.3 For all $a \in \mathbb{A}^\times$, $x \in k^\times$ and $\Phi \in \mathrm{Sch}_{00}(\mathbb{A})$ we have

$$\begin{pmatrix} ax \\ \Phi \end{pmatrix}_+ = \begin{pmatrix} a \\ \Phi^{(x)} \end{pmatrix}_+ = x^{\Theta(a,\Phi)} \begin{pmatrix} a \\ \Phi \end{pmatrix}_+ . \tag{66}$$

We note this for convenient reference. The proof is more or less the same as for (63).

8.4.4 The rational Fourier transform and the Catalan symbol do in fact depend on the choice of differential ω, even though we have suppressed reference to ω in the notation. To clarify this dependence let us temporarily write $\mathcal{F}_{0,\omega}$ and $\binom{\cdot}{\cdot}_\omega$. Then for all $t \in k^\times$, $\Phi \in \mathrm{Sch}(\mathbb{A})$ and $\xi \in \mathbb{A}$ we have

$$\mathcal{F}_{0,t\omega}[\Phi](\xi) = \mathcal{F}_{0,\omega}[\Phi](t\xi) = \mathcal{F}_{0,\omega}[\Phi]^{(t^{-1})}(\xi). \tag{67}$$

In turn, for all $a \in \mathbb{A}^\times$, $\Phi \in \mathrm{Sch}_{00}(\mathbb{A})$ and $t \in k^\times$, we have

$$\begin{pmatrix} a \\ \Phi \end{pmatrix}_{t\omega} = t^{-\Theta(a,\Phi)} \begin{pmatrix} a \\ \Phi \end{pmatrix}_{\omega} \tag{68}$$

by (66) above, functional equation (61) obeyed by the theta symbol, and the definitions.

8.4.5 Rewritten in terms of the Catalan symbol and the rational Fourier transform, with attention restricted to Schwartz functions taken from the group $\mathrm{Sch}_{00}(\mathbb{A})$, the adelic Stirling formula (53) takes a greatly simplified form, namely

$$\begin{aligned}
&\Theta(a,\Phi)\,\mathrm{ord}_v\,\omega + \mathrm{ord}_v \begin{pmatrix} a \\ \Phi \end{pmatrix} \\
= &\int \frac{\Theta(i_v^\times(t)a,\Phi) - \mathbf{1}_{\mathcal{O}_v^\times}(t)\Theta(a)}{\|1-t\|_v} d\mu_v^\times(t) \\
&+ \int \Phi(i_v(t))d\mu_v^\times(t) + \|a\| \int \tilde{\Phi}(i_v(t))d\mu_v^\times(t)
\end{aligned} \tag{69}$$

for all $a \in \mathbb{A}^\times$, $\Phi \in \mathrm{Sch}_{00}(\mathbb{A})$ and places v of k. Clearly, the right side of (69) is independent of the differential ω. As a consistency check, note that (68) forces the left side of (69) to be independent of ω.

Theorem 8.5. *Fix*

$$x_0 \in \mathbb{A}, \quad a,b \in \mathbb{A}^\times, \quad f \in \mathbb{A}^\times \cap \mathcal{O}$$

such that

$$x_0 \notin b\mathcal{O}, \quad fx_0 \in b\mathcal{O}, \quad \left\{\begin{array}{ccc} \|ab\| & > & q^{2g-2} \\ & \text{or} & \\ \|ab\| & < & \|f\|. \end{array}\right.$$

Then the map

$$\left(x + b\mathcal{O} \mapsto \left\{\begin{array}{ll} \left(\begin{array}{c} a \\ \mathbf{1}_{x+b\mathcal{O}} - \mathbf{1}_{x_0+b\mathcal{O}} \end{array}\right) & \text{if } x \notin b\mathcal{O} \\ 0 & \text{if } x \in b\mathcal{O} \end{array}\right.\right) : f^{-1}b\mathcal{O}/b\mathcal{O} \to k_{\mathrm{perf}}$$

is $\mathbb{F}_q$-linear.

Thus, besides obvious $\mathbb{Z}[1/q]$-linearity ($\mathrm{Sch}_0(\mathbb{A})$ to $k_{\mathrm{perf}}^\times$), the Catalan symbol has a "hidden" $\mathbb{F}_q$-linearity. As the proof shows, the theorem is essentially just a rehash of Propositions 3.6 and 3.8.

Proof. By scaling rule (64), we may assume without loss of generality that $a = 1$. Let γ_{b,f,x_0} denote the map in question (with $a = 1$). Our task is to prove that γ_{b,f,x_0} is $\mathbb{F}_q$-linear. To do so, we distinguish two cases, namely: (i) $\|b\| > q^{2g-2}$ and (ii) $\|b\| < \|f\|$.

We turn to case (i). Put

$$V = f^{-1}b\mathcal{O} \cap k, \quad W = b\mathcal{O} \cap k.$$

For all $x \in f^{-1}b\mathcal{O}$, the quantity $\#((x + b\mathcal{O}) \cap k) = \Theta(1, \mathbf{1}_{x+b\mathcal{O}})$ is positive and independent of x by Proposition 7.6. It follows that the inclusion-induced natural map $V/W \to f^{-1}b\mathcal{O}/b\mathcal{O}$ is bijective. It suffices to prove the $\mathbb{F}_q$-linearity of the map $\tilde{\gamma}_{b,f,x_0}$ obtained by following the isomorphism $V/W \to f^{-1}b\mathcal{O}/b\mathcal{O}$ by γ_{b,f,x_0}. Choose $x_1 \in V$ such that $x_0 + b\mathcal{O} = x_1 + b\mathcal{O}$. By Proposition 8.3 and the definition of the toy Catalan symbol we have

$$\tilde{\gamma}_{b,f,x_0}(x + W) = \left(\begin{array}{c} 1 \\ \mathbf{1}_{x+b\mathcal{O}} - \mathbf{1}_{x_0+b\mathcal{O}} \end{array}\right)_+ = \left(\begin{array}{c} \alpha \\ \mathbf{1}_{x+W} \end{array}\right) \Big/ \left(\begin{array}{c} \alpha \\ \mathbf{1}_{x_1+W} \end{array}\right)$$

for all $x \in V \setminus W$, where α is the inclusion $V \to k_{\mathrm{perf}}$. Therefore the map $\tilde{\gamma}_{b,f,x_0}$ is $\mathbb{F}_q$-linear by Proposition 3.6. Thus case (i) is proved.

We turn to case (ii). In the obvious way let us now identify the space $\mathrm{Sch}(f^{-1}b\mathcal{O}/b\mathcal{O})$ of complex-valued functions on the finite set $f^{-1}b\mathcal{O}/b\mathcal{O}$ with the subspace of $\mathrm{Sch}(\mathbb{A})$ consisting of functions constant on cosets of $b\mathcal{O}$ and supported in $f^{-1}b\mathcal{O}$. By example (58) and the remark following we have

$$\mathcal{F}_0[\mathrm{Sch}(f^{-1}b\mathcal{O}/b\mathcal{O})] \subset \mathrm{Sch}(b^{-1}\kappa^{-1}\mathcal{O}/fb^{-1}\kappa^{-1}\mathcal{O}).$$

But the spaces $f^{-1}b\mathcal{O}/b\mathcal{O}$ and $b^{-1}\kappa^{-1}\mathcal{O}/fb^{-1}\kappa^{-1}\mathcal{O}$ are Serre dual with respect to the pairing $\langle\cdot,\cdot\rangle$ and so we also have at our disposal a toy rational Fourier transform

$$\mathrm{Sch}(f^{-1}b\mathcal{O}/b\mathcal{O}) \xrightarrow{\mathcal{F}_0^{\mathrm{toy}}} \mathrm{Sch}(b^{-1}\kappa^{-1}\mathcal{O}/fb^{-1}\kappa^{-1}\mathcal{O}).$$

By comparing the examples (11) and (58) it can be seen that there exists an integer ℓ such that

$$q^\ell\, \mathcal{F}_0^{\mathrm{toy}}[\Phi] = \mathcal{F}_0[\Phi]$$

for all $\Phi \in \mathrm{Sch}(f^{-1}b\mathcal{O}/b\mathcal{O}) \subset \mathrm{Sch}(\mathbb{A})$. Fix $\xi_0 \in b^{-1}\kappa^{-1}\mathcal{O} \setminus fb^{-1}\kappa^{-1}\mathcal{O}$ arbitrarily. By case (i) already proved, the map

$$\gamma_{b^{-1}\kappa^{-1}f\mathcal{O},f,\xi_0} : b^{-1}\kappa^{-1}\mathcal{O}/fb^{-1}\kappa^{-1}\mathcal{O} \to k_{\mathrm{perf}}$$

is $\mathbb{F}_q$-linear. Let β be the $\mathbb{F}_q$-linear map obtained by multiplying $\gamma_{b^{-1}\kappa^{-1}f\mathcal{O},f,\xi_0}$ by the factor $\left(\begin{array}{c}1\\ \mathbf{1}_{\xi_0+fb^{-1}\kappa^{-1}\mathcal{O}}\end{array}\right)_+$. By Proposition 8.3, the map β takes the form

$$\xi+fb^{-1}\kappa^{-1}\mathcal{O}\mapsto\left\{\begin{array}{ll}\left(\begin{array}{c}1\\ \mathbf{1}_{\xi+fb^{-1}\kappa^{-1}\mathcal{O}}\end{array}\right)_+ & \text{if } \xi\not\in fb^{-1}\kappa^{-1}\mathcal{O},\\ 0 & \text{otherwise.}\end{array}\right.$$

Note that β is injective. By Proposition 8.3 and the definition of the toy Catalan symbol we have

$$\begin{aligned}\gamma_{b,f,x_0}(x+b\mathcal{O}) &= \left(\begin{array}{c}1\\ \mathcal{F}_0[\mathbf{1}_{x+b\mathcal{O}}-\mathbf{1}_{x_0+b\mathcal{O}}]\end{array}\right)_+\\ &= \left(\begin{array}{c}\beta\\ \mathcal{F}_0^{\mathrm{toy}}[\mathbf{1}_{x+b\mathcal{O}}-\mathbf{1}_{x_0+b\mathcal{O}}]\end{array}\right)^{q^\ell}\end{aligned}$$

for all $x\in f^{-1}b\mathcal{O}\setminus b\mathcal{O}$. Therefore the map γ_{b,f,x_0} is $\mathbb{F}_q$-linear by Proposition 3.8. Thus case (ii) is proved. $\square$

9. Formulation of a conjecture

In this section we state the conjecture (Conjecture 9.5) to which we allude in the title of the paper. The conjecture links theta and Catalan symbols to two-variable algebraic functions of a certain special type. Following the conjecture we make some amplifying remarks.

9.1 The ring k

9.1.1 Put

$$\mathbf{k}_0=k\otimes_{\mathbb{F}_q}k,\quad \Delta_0=\ker\left((x\otimes y\mapsto xy):k\otimes_{\mathbb{F}_q}k\to k\right).$$

It is not difficult to verify that the ring $\mathbf{k}_0$ is a noetherian integrally closed domain of dimension one, and hence a Dedekind domain. Clearly, the ideal Δ_0 is maximal in $\mathbf{k}_0$, and by construction has residue field canonically identified with k. Let $\hat{\mathbf{k}}_0$ be the completion of $\mathbf{k}_0$ with respect to Δ_0, and let $\hat{\Delta}_0$ be the closure of Δ_0 in $\hat{\mathbf{k}}_0$. Then $\hat{\mathbf{k}}_0$ is a discrete valuation ring with residue field canonically identified with k and with maximal ideal $\hat{\Delta}_0$. Put

$$\mathbf{k}=(\hat{\mathbf{k}}_0)_{\mathrm{perf}},\quad \Delta=\bigcup\sqrt[q^n]{\hat{\Delta}_0}.$$

Then $\mathbf{k}$ is a (nondiscrete) valuation ring with residue field canonically identified with k_{perf} and with maximal ideal Δ.

9.1.2 We identify the universal derivation $d:k\to\Omega$ with the map

$$(x\mapsto x\otimes 1-1\otimes x \bmod \Delta_0^2):k\to\Delta_0/\Delta_0^2=\hat{\Delta}_0/\hat{\Delta}_0^2,$$

thus fixing an identification

$$\Omega=\hat{\Delta}_0/\hat{\Delta}_0^2.$$

We fix a generator ϖ of the principal ideal $\hat{\Delta}_0$ such that

$$\omega\equiv\varpi \bmod \hat{\Delta}_0^2.$$

We call ϖ a *lifting* of ω.

9.1.3 Every nonzero element φ of the fraction field of $\hat{\mathbf{k}}_0$ has a unique $\hat{\Delta}_0$-adic expansion of the form

$$\varphi = \sum_{i=i_0}^{\infty} (1 \otimes a_i)\varpi^i \quad (i_0 \in \mathbb{Z},\ \ a_i \in k,\ \ a_{i_0} \neq 0)$$

and in terms of this expansion we define

$$\mathrm{ord}_\Delta\, \varphi = i_0, \quad \mathrm{lead}_\Delta = a_{i_0}.$$

Also put $\mathrm{ord}_\Delta\, 0 = +\infty$. More generally, for all nonzero φ in the fraction field of $\mathbf{k}$, we define

$$\mathrm{ord}_\Delta\, \varphi = q^{-n}\,\mathrm{ord}_\Delta\, \varphi^{q^n} \in \mathbb{Z}[1/q], \quad \mathrm{lead}_\Delta\, \varphi = (\mathrm{lead}_\Delta\, \varphi^{q^n})^{q^{-n}} \in k_{\mathrm{perf}}^\times$$

for all sufficiently large n. The function ord_Δ is an additive valuation of the fraction field of $\mathbf{k}$ independent of the choice of differential ω and lifting ϖ. The function lead_Δ is a homomorphism from the multiplicative group of the fraction field of $\mathbf{k}$ to $k_{\mathrm{perf}}^\times$ depending only on the differential ω, not on the lifting ϖ.

9.1.4 We clarify the dependence of lead_Δ on the choice of differential ω as follows. Let us temporarily write $\mathrm{lead}_{\Delta,\omega}$ to stress the ω-dependence. Then we have

$$\mathrm{lead}_{\Delta,t\omega}\, \varphi = t^{-\mathrm{ord}_\Delta \varphi}\, \mathrm{lead}_{\Delta,\omega}\, \varphi \tag{70}$$

for all $t \in k^\times$ and nonzero elements φ of the fraction field of $\mathbf{k}$. This ought to be compared with equation (68) above.

9.1.5 The exchange-of-factors automorphism

$$(x \otimes y \mapsto y \otimes x) : \mathbf{k}_0 \to \mathbf{k}_0$$

preserves Δ_0, hence has a unique $\hat{\Delta}_0$-adically continuous extension to an automorphism of $\hat{\mathbf{k}}_0$, and hence has a unique extension to an automorphism of the fraction field of $\mathbf{k}$. The result of applying the latter automorphism to an element φ of the fraction field of $\mathbf{k}$ we denote by $\varphi^\dagger$. We have

$$(\varphi^\dagger)^\dagger = \varphi, \quad \mathrm{ord}_\Delta\, \varphi^\dagger = \mathrm{ord}_\Delta\, \varphi, \quad \mathrm{lead}_\Delta\, \varphi^\dagger = (-1)^{\mathrm{ord}_\Delta \varphi}\, \mathrm{lead}_\Delta\, \varphi \tag{71}$$

for all nonzero φ in the fraction field of $\mathbf{k}$.

9.2 The ring K

9.2.1 For any abelian extension K/k, let

$$\rho = \rho_{K/k} : \mathbb{A}^\times \to \mathrm{Gal}(K/k)$$

be the reciprocity law homomorphism of global class field theory defined according to the nowadays commonly followed convention of Tate's paper [Ta79]. It is important to bear in mind two facts concerning this convention, which differs from the older convention of, say, Artin-Tate [AT]. Firstly, if v is a place of k unramified in K, and $\tau \in k_v^\times$ is a uniformizer at v, then $\rho_{K/k}(i_v^\times(\tau)) \in \mathrm{Gal}(K/k)$ is a geometric Frobenius element (inverse of the usual Artin symbol) at v. Secondly, we have

$$C^{\rho_{K/k}(a)} = C^{\|a\|}$$

for all $a \in \mathbb{A}^\times$ and constants $C \in K$.

9.2.2 Let k^{ab} be the abelian closure of k in the algebraic closure $\bar{k}$. Let $\mathbb{F}_q^{\mathrm{ab}}$ be the abelian (also the algebraic) closure of $\mathbb{F}_q$ in $\bar{k}$. Note that $\mathrm{Gal}(k^{\mathrm{ab}}/k) = \mathrm{Gal}(k_{\mathrm{perf}}^{\mathrm{ab}}/k_{\mathrm{perf}})$ and that the natural

action of $\mathrm{Gal}(k^{\mathrm{ab}}/k)$ on $k^{\mathrm{ab}}_{\mathrm{perf}}$ commutes with the q^{th} power automorphism of $k^{\mathrm{ab}}_{\mathrm{perf}}$. The group

$$\left\{[\sigma,\tau]\in \mathrm{Gal}(k^{\mathrm{ab}}/k)\times \mathrm{Gal}(k^{\mathrm{ab}}/k)\,\middle|\,\sigma|_{\mathbb{F}_q^{\mathrm{ab}}}=\tau|_{\mathbb{F}_q^{\mathrm{ab}}}\right\}$$

acts naturally on the integral domains

$$k^{\mathrm{ab}}\otimes_{\mathbb{F}_q^{\mathrm{ab}}} k^{\mathrm{ab}},\quad k^{\mathrm{ab}}_{\mathrm{perf}}\otimes_{\mathbb{F}_q^{\mathrm{ab}}} k^{\mathrm{ab}}_{\mathrm{perf}}$$

by the rule

$$(x\otimes y)^{[\sigma,\tau]}=x^\sigma\otimes y^\tau,$$

with fixed rings $\mathbf{k}_0$ and $\mathbf{k}$, respectively. Put

$$\delta\,\mathrm{Gal}(k^{\mathrm{ab}}/k)=\{[\sigma,\sigma]|\sigma\in \mathrm{Gal}(k^{\mathrm{ab}}/k)\},$$

$$\mathbf{K}_0=(k^{\mathrm{ab}}\otimes_{\mathbb{F}_q^{\mathrm{ab}}} k^{\mathrm{ab}})^{\delta\,\mathrm{Gal}(k^{\mathrm{ab}}/k)},\quad \mathbf{K}=(k^{\mathrm{ab}}_{\mathrm{perf}}\otimes_{\mathbb{F}_q^{\mathrm{ab}}} k^{\mathrm{ab}}_{\mathrm{perf}})^{\delta\,\mathrm{Gal}(k^{\mathrm{ab}}/k)}.$$

Note that $\mathbf{K}=(\mathbf{K}_0)_{\mathrm{perf}}$. Note that $\mathrm{Gal}(k^{\mathrm{ab}}/\mathbb{F}_q^{\mathrm{ab}})$ may be identified with the Galois groups of the etale ring extensions $\mathbf{K}_0/\mathbf{k}_0$ and $\mathbf{K}/\mathbf{k}$ under the map $\sigma\mapsto[\sigma,1]$.

9.2.3 Let $a\in\mathbb{A}^\times$ be given. The automorphism

$$\left(x\otimes y\mapsto x^{\rho(a)}\otimes y^{\|a\|}\right): k^{\mathrm{ab}}_{\mathrm{perf}}\otimes_{\mathbb{F}_q^{\mathrm{ab}}} k^{\mathrm{ab}}_{\mathrm{perf}}\rightarrow k^{\mathrm{ab}}_{\mathrm{perf}}\otimes_{\mathbb{F}_q^{\mathrm{ab}}} k^{\mathrm{ab}}_{\mathrm{perf}}$$

commutes with the action of the group $\delta\,\mathrm{Gal}(k^{\mathrm{ab}}/k)$, hence descends to an automorphism of $\mathbf{K}$ and hence has a unique extension to an automorphism of the fraction field of $\mathbf{K}$. The result of applying the latter automorphism to an element φ of the fraction field of $\mathbf{K}$ we denote by $\varphi^{(a)}$. Thus we equip the fraction field of $\mathbf{K}$ with an action of $\mathbb{A}^\times$ factoring through the idele class group $\mathbb{A}^\times/k^\times$.

9.2.4 The exchange-of-factors automorphism

$$(x\otimes y\mapsto y\otimes x): k^{\mathrm{ab}}_{\mathrm{perf}}\otimes_{\overline{\mathbb{F}}_q} k^{\mathrm{ab}}_{\mathrm{perf}}\rightarrow k^{\mathrm{ab}}_{\mathrm{perf}}\otimes_{\overline{\mathbb{F}}_q} k^{\mathrm{ab}}_{\mathrm{perf}}$$

commutes with the action of the group $\delta\,\mathrm{Gal}(k^{\mathrm{ab}}/k)$, hence descends to an automorphism of $\mathbf{K}$, and hence has a unique extension to an automorphism of the fraction field of $\mathbf{K}$. The result of applying the latter automorphism to an element φ of the fraction field of $\mathbf{K}$ we denote by $\varphi^\dagger$. We have

$$(\varphi^\dagger)^{(a)}=((\varphi^{(a^{-1})})^\dagger)^{\|a\|}\tag{72}$$

for all φ in the fraction field of $\mathbf{K}$ and $a\in\mathbb{A}^\times$.

Proposition 9.3. *(i) There exists a unique $\mathbf{k}_0$-algebra embedding*

$$\iota:\mathbf{K}\rightarrow\mathbf{k}$$

such that

$$\iota^{-1}(\Delta)=\mathbf{K}\cap\ker\left((x\otimes y\mapsto xy): k^{\mathrm{ab}}_{\mathrm{perf}}\otimes_{\mathbb{F}_q^{\mathrm{ab}}} k^{\mathrm{ab}}_{\mathrm{perf}}\rightarrow k^{\mathrm{ab}}_{\mathrm{perf}}\right).$$

(ii) Moreover, ι commutes with $\dagger$.

After the proof of this proposition, in order to keep notation simple, we just identify $\mathbf{K}$ with its image in $\mathbf{k}$ under ι, and dispense with the notation ι altogether. In this way $\mathrm{ord}_\Delta\,\varphi$ and $\mathrm{lead}_\Delta\,\varphi$ are defined for all nonzero φ in the fraction field of $\mathbf{K}$, and moreover both (71) and (72) hold.

Proof. Part (i) granted, the $\mathbf{k}_0$-algebra embedding $\varphi\mapsto(\iota(\varphi^\dagger))^\dagger$ has the property uniquely characterizing ι and hence coincides with ι. It is enough to prove part (i). In turn, it is enough to prove that there exists a unique $\mathbf{k}_0$-algebra embedding $\iota_0:\mathbf{K}_0\rightarrow\hat{\mathbf{k}}_0$ such that

$$\iota_0^{-1}(\hat{\Delta}_0)=\mathbf{K}_0\cap\ker\left((x\otimes y\mapsto xy): k^{\mathrm{ab}}\otimes_{\mathbb{F}_q^{\mathrm{ab}}} k^{\mathrm{ab}}\rightarrow k^{\mathrm{ab}}\right),$$

because once this is proved, the induced homomorphism

$$(\iota_0)_{\mathrm{perf}} : (\mathbf{K}_0)_{\mathrm{perf}} \to (\hat{\mathbf{k}}_0)_{\mathrm{perf}}$$

is the only possibility for ι.

Now let K/k be any finite abelian extension and consider the following naturally associated objects:

- Let $G = \mathrm{Gal}(K/k)$.
- Let $\mathbb{F}$ be the constant field of K.
- Let $G' = \{[\sigma,\tau] \in G \times G | \sigma|_{\mathbb{F}} = \tau|_{\mathbb{F}}\}$.
- Let G' act on $K \otimes_{\mathbb{F}} K$ by the rule $(x \otimes y)^{[\sigma,\tau]} = x^\sigma \otimes y^\tau$.
- Let $\delta G = \{[\sigma,\sigma] | \sigma \in G\} \subset G'$.
- Put $R = (K \otimes_{\mathbb{F}} K)^{\delta G}$.
- Put $I = R \cap \ker\left((x \otimes y \mapsto xy) : K \otimes_{\mathbb{F}} K \to K\right)$.

It is enough to prove that there exists a unique $\mathbf{k}_0$-algebra homomorphism $\iota_{K/k} : R \to \hat{\mathbf{k}}_0$ such that $\iota_{K/k}^{-1}(\hat{\Delta}_0) = I$, because once this is proved the limit $\lim_{\to} \iota_{K/k} : \mathbf{K}_0 \to \hat{\mathbf{k}}_0$ extended over all finite subextensions K/k of k^{ab}/k is the only possibility for ι_0. Now the ring $K \otimes_{\mathbb{F}} K$ is a Dedekind domain finite etale and abelian over $\mathbf{k}_0$. It follows that the δG-fixed subring R has these same properties by descent. The prime $I \subset R$ lies above the prime $\Delta_0 \subset \mathbf{k}_0$ and we have $R/I = \mathbf{k}_0/\Delta_0 = k$. Since R is etale over $\mathbf{k}_0$, existence and uniqueness of $\iota_{K/k}$ follow now by the infinitesimal lifting criterion. ☐

9.4 Coleman units

We declare a nonzero element φ of the fraction field of $\mathbf{K}$ to be a *Coleman unit* if for every maximal ideal $M \subset \mathbf{K}$ not of the form $\{\psi \in \mathbf{K} | \operatorname{ord}_\Delta \psi^{(a)} > 0\}$ for some $a \in \mathbb{A}^\times$, we have $\varphi \in \mathbf{K}_M^\times$, where $\mathbf{K}_M$ is the local ring of $\mathbf{K}$ at M. The Coleman units form a group under multiplication. For every Coleman unit φ and $a \in \mathbb{A}^\times$, again $\varphi^{(a)}$ is a Coleman unit, and so also is $\varphi^\dagger$. The group of Coleman units is closed under the extraction of q^{th} roots. The inspiration for this definition comes from Coleman's paper [Co88].

Conjecture 9.5. *For every $\Phi \in \mathrm{Sch}_{00}(\mathbb{A})$, there exists a unique Coleman unit φ such that*

$$\operatorname{ord}_\Delta \varphi^{(a)} = \Theta(a, \Phi), \quad \operatorname{lead}_\Delta \varphi^{(a)} = \left(\begin{array}{c} a \\ \Phi \end{array}\right) \tag{73}$$

for all $a \in \mathbb{A}^\times$.

9.6 Amplification

9.6.1 We have, so we claim, the following *uniqueness principle*: for every ψ in the fraction field of $\mathbf{K}$,

$$\#\{\|a\| \mid a \in \mathbb{A}^\times,\ \operatorname{ord}_\Delta \psi^{(a)} \neq 0\} = \infty \Rightarrow \psi = 0. \tag{74}$$

In any case, we can find a Dedekind domain R between $\mathbf{K}$ and $\mathbf{k}_0$ which is finite over $\mathbf{k}_0$ and to which ψ belongs. Since the maximal ideals of $\mathbf{k}_0$ of the form

$$\ker\left((x \otimes y \mapsto xy^{q^n}) : \mathbf{k}_0 \to k_{\mathrm{perf}}\right) \quad (n \in \mathbb{Z})$$

are distinct, the hypothesis implies that ψ has nonzero valuation at infinitely many maximal ideals of R and hence vanishes identically. The claim is proved. By the uniqueness principle, for each $\Phi \in \mathrm{Sch}_{00}(\mathbb{A})$, there can be at most one element φ of the fraction field of $\mathbf{K}$ satisfying (73) for all $a \in \mathbb{A}^\times$. So the uniqueness asserted in the conjecture is clear. Only existence is at issue.

9.6.2 Let nonzero φ in the fraction field of $\mathbf{K}$, $\Phi \in \mathrm{Sch}_{00}(\mathbb{A})$, and $a \in \mathbb{A}^\times$ be given. We write $\varphi \sim_a \mathcal{C}[\Phi]$ if (73) holds for the data φ, a and Φ. We write $\varphi = \mathcal{C}[\Phi]$ if $\varphi \sim_a \mathcal{C}[\Phi]$ for all $a \in \mathbb{A}^\times$ and φ is a Coleman unit. In the situation $\varphi = \mathcal{C}[\Phi]$, as the notation is meant to suggest, we think of φ as the image of Φ under a transformation $\mathcal{C}$. From the latter point of view, the conjecture asserts the well-definedness of this transformation $\mathcal{C}$ on the whole of the space $\mathrm{Sch}_{00}(\mathbb{A})$.

9.6.3 Let φ in the fraction field of $\mathbf{K}$ and $\Phi \in \mathrm{Sch}_{00}(\mathbb{A})$ be given. Comparison with (73) of the formula (68) explaining the dependence of $(\dot{\cdot})$ on ω and the formula (70) explaining the dependence of lead_Δ on ω shows that if $\varphi = \mathcal{C}[\Phi]$, then no matter what nonzero differential $\omega \in \Omega_{k/\mathbb{F}_q}$ we choose to define rational Fourier transforms, Catalan symbols, and "leading Taylor coefficients at Δ", it remains the case that $\varphi = \mathcal{C}[\Phi]$. In short, $\mathcal{C}$ is independent of ω. So if the conjecture is true for one choice of ω, it is true for all.

9.6.4 Grant the conjecture for this paragraph, so that the transformation $\mathcal{C}$ is defined. Clearly, we have

$$\mathcal{C}[n_1\Phi_1 + n_2\Phi_2] = \mathcal{C}[\Phi_1]^{n_1}\mathcal{C}[\Phi_2]^{n_2} \tag{75}$$

for all $\Phi_1, \Phi_2 \in \mathrm{Sch}_{00}(\mathbb{A})$ and $n_1, n_2 \in \mathbb{Z}[1/q]$. Comparison with (73) of the scaling rule (50) for the theta symbol and the scaling rule (64) for the Catalan symbol shows that

$$\mathcal{C}[\Phi^{(a)}] = \mathcal{C}[\Phi]^{(a)} \tag{76}$$

for all $a \in \mathbb{A}^\times$ and $\Phi \in \mathrm{Sch}_{00}(\mathbb{A})$. Comparison with (73) of the functional equation (61) for the theta symbol, the functional equation (65) for the Catalan symbol, and the properties (71) and (72) of the dagger operation on the fraction field of $\mathbf{K}$ shows that

$$\mathcal{C}[\tilde{\Phi}] = \mathcal{C}[\Phi]^\dagger \tag{77}$$

for all $\Phi \in \mathrm{Sch}_{00}(\mathbb{A})$. For any fixed $b \in \mathbb{A}^\times$ and $x_0 \in \mathbb{A} \setminus b\mathcal{O}$ the map

$$\left(x + b\mathcal{O} \mapsto \left\{ \begin{array}{rl} \mathcal{C}\left[\mathbf{1}_{x+b\mathcal{O}} - \mathbf{1}_{x_0+b\mathcal{O}}\right] & \text{if } x \notin b\mathcal{O} \\ 0 & \text{if } x \in b\mathcal{O} \end{array}\right.\right) : \mathbb{A}/b\mathcal{O} \to \mathbf{K} \tag{78}$$

is $\mathbb{F}_q$-linear by the "hidden" $\mathbb{F}_q$-linearity of the Catalan symbol (Theorem 8.5) and the uniqueness principle (74).

9.6.5 We return now to the general discussion of the conjecture, no longer assuming that it holds. It is easy to verify that the set of $\Phi \in \mathrm{Sch}_{00}(\mathbb{A})$ for which $\mathcal{C}[\Phi]$ is defined is a $\mathbb{Z}[1/q][\mathbb{A}^\times]$-submodule of $\mathrm{Sch}_{00}(\mathbb{A})$. It follows that to prove the conjecture it is enough to fix a set of generators for $\mathrm{Sch}_{00}(\mathbb{A})$ as a $\mathbb{Z}[1/q][\mathbb{A}^\times]$-module and for each generator Φ to prove the existence of $\mathcal{C}[\Phi]$. For example, for any fixed $b \in \mathbb{A}^\times$ and $x_0 \in \mathbb{A} \setminus b\mathcal{O}$, the family $\{\mathbf{1}_{x+b\mathcal{O}} - \mathbf{1}_{x_0+b\mathcal{O}} | x \in \mathbb{A} \setminus b\mathcal{O}\}$ is large enough to generate $\mathrm{Sch}_{00}(\mathbb{A})$ as a $\mathbb{Z}[1/q][\mathbb{A}^\times]$-module.

9.6.6 Here are some trivial but handy cases in which we can easily prove that the transformation $\mathcal{C}$ is defined. Fix $\Phi \in \mathrm{Sch}_0(\mathbb{A})$. Fix $x \in k^\times$. Note that $\Phi^{(x)} - \Phi \in \mathrm{Sch}_{00}(\mathbb{A})$. We claim that

$$\mathcal{C}[\Phi^{(x)} - \Phi] = x^{-\Phi(0)} \otimes x^{\tilde{\Phi}(0)}. \tag{79}$$

In any case, the right side is a unit of $\mathbf{K}$ and *a fortiori* a Coleman unit; further, we have

$$\Theta(a, \Phi^{(x)} - \Phi) = 0$$

for all $a \in \mathbb{A}^\times$ by (49) and (50); and finally we have

$$\left(\begin{array}{c} a \\ \Phi^{(x)} - \Phi \end{array}\right) = x^{-\Phi(0)+\|a\|\tilde{\Phi}(0)}$$

for all $a \in \mathbb{A}^\times$ by (63) and (64). This is enough to prove the claim.

10. Proof of the conjecture in genus zero

We assume in §10 that $g = 0$ and under this additional assumption we are going to prove Conjecture 9.5.

10.1 Notation and reductions

Every function field of genus zero with field of constants $\mathbb{F}_q$ has a place with residue field $\mathbb{F}_q$, and as we noted in §9.6, the conjecture is invariant under change of differential. We therefore may assume without loss of generality that

$$k = \mathbb{F}_q(T), \quad \omega = dT,$$

where T is transcendental over $\mathbb{F}_q$. We take

$$\varpi = T \otimes 1 - 1 \otimes T \in \mathbf{k}_0$$

as a lifting of ω. Note that ϖ is already a prime element of the principal ideal domain $\mathbf{k}_0$. Note also that ϖ is a Coleman unit. Let ∞ be the unique place of k at which T has a pole and put

$$\tau = i_\infty^\times(T^{-1}) \in \mathbb{A}^\times,$$

noting that

$$\|\tau\| = q^{-1}.$$

Put

$$\mathcal{U} = \left\{ a = [a_v] \in \mathcal{O}^\times \mid \|a_\infty - 1\|_\infty < 1 \right\}.$$

Then every $a \in \mathbb{A}^\times$ factors uniquely in the form

$$a = uz\tau^{-N} \quad (u \in \mathcal{U},\ z \in k^\times,\ N \in \mathbb{Z}).$$

We use this factorization repeatedly below. For each $x \in k$, write

$$x = \lfloor x \rfloor + \langle x \rangle \quad (\lfloor x \rfloor \in \mathbb{F}_q[T], \ \ \|\langle x \rangle\|_\infty < 1)$$

in the unique possible way, in parallel with the definitions made in §4.1.5. Put

$$\deg a = -\operatorname{ord}_\infty a = (\text{degree of } a \text{ as a polynomial in } T)$$

for $a \in \mathbb{F}_q[T]$. For each $x \in k^\times$, put

$$\Psi_x = \mathbf{1}_{x+\tau\mathcal{O}} - \mathbf{1}_{1+\tau\mathcal{O}} \in \mathrm{Sch}_{00}(\mathbb{A}).$$

The set $\{\Psi_x\}$ generates $\mathrm{Sch}_{00}(\mathbb{A})$ as a $\mathbb{Z}[1/q][\mathbb{A}^\times]$-module, and so to prove the conjecture in the case at hand it suffices to prove that $\mathcal{C}[\Psi_x]$ exists for every $x \in k^\times$.

10.2 Calculation of theta and Catalan symbols

Let $N \in \mathbb{Z}$ and $x \in k^\times$ be given. We calculate $\Theta(\tau^{-N}, \Psi_x)$ and $\left(\begin{array}{c} \tau^{-N} \\ \Psi_x \end{array}\right)$.

10.2.1 Assume at first that $N \geqslant 0$, which is the relatively easy case. We shall handle the case $N < 0$ presently. We have

$$\begin{array}{rcl} k \cap \tau^{-N}(x+\tau\mathcal{O}) & = & \{z \in k |\ \langle x\rangle = \langle z\rangle,\ \ \lfloor x\rfloor = \lfloor T^{-N}z\rfloor\} \\ & = & \langle x\rangle + T^N\lfloor x\rfloor + \{a \in \mathbb{F}_q[T]|\deg a < N\}. \end{array}$$

Therefore we have

$$\Theta(\tau^{-N},\Psi_x) = 0, \tag{80}$$

$$\left(\begin{array}{c}\tau^{-N}\\ \Psi_x\end{array}\right) = \left(\begin{array}{c}\tau^{-N}\\ \Psi_x\end{array}\right)_+ = \prod_{\substack{a\in\mathbb{F}_q[T]\\ \deg a<N}} \frac{a+\langle x\rangle + T^N\lfloor x\rfloor}{a+T^N}, \tag{81}$$

by Proposition 8.3 at the first equality of (81), and at the other two equalities by direct appeal to the definitions.

10.2.2 We now take a close look at the rational Fourier transform $\tilde{\Psi}_x$ in order to prepare for handling the case $N < 0$. By specializing the example (58) of an adelic rational Fourier transform to the present case (we may take $\kappa = \tau^{-2}$ because $\omega = dT$ has a double pole at ∞ and no other poles or zeroes), we have

$$\mathcal{F}_0[\mathbf{1}_{x+\tau\mathcal{O}}] = \mathbf{1}_{\tau\mathcal{O}\cap[\langle x,\cdot\rangle=0]} - \mathbf{1}_{\tau\mathcal{O}\cap[\langle x,\cdot\rangle=1]}$$

and hence

$$\tilde{\Psi}_x = (1+\mathcal{N})\,[\Psi_x^\star] \quad (\Psi_x^\star = -\mathbf{1}_{\tau\mathcal{O}\cap[\langle x,\cdot\rangle=1]} + \mathbf{1}_{\tau\mathcal{O}\cap[\langle 1,\cdot\rangle=1]}), \tag{82}$$

where to get (82) from the preceding formula we reused the "starred" trick from the proof of Proposition 3.8. It follows (without any restriction on N) that

$$\Theta(\tau^{-N},\Psi_x) = q^N\Theta(\tau^N,\tilde{\Psi}_x) = q^{N+1}\Theta(\tau^N,\Psi_x^\star), \tag{83}$$

by functional equation (61) at the first equality, and scaling rule (50) at the second. It follows in turn (again without restriction on N) that

$$\left(\begin{array}{c}\tau^{N}\\ \tilde{\Psi}_x\end{array}\right)_+ = (-1)^{\Theta(\tau^{-N},\Psi_x)}\left(\begin{array}{c}\tau^{N}\\ \Psi_x^\star\end{array}\right)_+^q, \tag{84}$$

by scaling rule (66) followed by an application of (83).

10.2.3 We now handle the remaining case $N < 0$ of our calculation. Put

$$\nu = -N-1,$$

noting that $\nu \geqslant 0$. Put

$$V = \{a \in \mathbb{F}_q[T]|\deg a \leqslant \nu\} = \tau^{N+1}\mathcal{O}\cap k,$$

$$\left.\begin{array}{rcl} \xi & = & (a \mapsto -\langle x, \tau^{-N}a\rangle) \\ \xi_1 & = & (a\mapsto -\langle 1,\tau^{-N}a\rangle) \end{array}\right\} : V \to \mathbb{F}_q,$$

noting that

$$\xi(a) = \mathrm{Res}_\infty(a(\langle x\rangle - T^{-\nu-1}\lfloor x\rfloor)\,dT)),\ \ \xi_1(a) = -\,\mathrm{Res}_\infty(aT^{-\nu-1}\,dT).$$

The latter relations are verified by applying "sum of residues equals zero" and then discarding terms not contributing to the residue at $T=\infty$. Since

$$k\cap\tau^N(\tau\mathcal{O}\cap[\langle x,\cdot\rangle = 1]) = \{a\in V|\xi(a) = -1\}, \tag{85}$$

we have via (83) that

$$\begin{aligned}&\Theta(\tau^{\nu+1},\Psi_x)\\ =&\left\{\begin{array}{ll}0&\text{if }\xi\neq 0,\\ 1&\text{if }\xi=0,\end{array}\right.\\ =&\left\{\begin{array}{ll}0&\text{if }\langle x\rangle-T^{-\nu-1}\lfloor x\rfloor\not\in T^{-\nu-2}\mathbb{F}_q[[1/T]]+\mathbb{F}_q[T],\\ 1&\text{if }\langle x\rangle-T^{-\nu-1}\lfloor x\rfloor\in T^{-\nu-2}\mathbb{F}_q[[1/T]]+\mathbb{F}_q[T].\end{array}\right.\end{aligned}\tag{86}$$

It follows that

$$\begin{aligned}&k\cap\tau^{-N}(x+\tau\mathcal{O})\\ =&\{z\in k|\ \langle x\rangle=\langle z\rangle,\ \ \lfloor x\rfloor=\lfloor T^{\nu+1}z\rfloor\}\\ =&\left\{\begin{array}{ll}\emptyset&\text{if }\Theta(\tau^{-N},\Psi_x)=0,\\ \{\langle x\rangle+\lfloor T^{-\nu-1}x\rfloor\}&\text{if }\Theta(\tau^{-N},\Psi_x)=1.\end{array}\right.\end{aligned}\tag{87}$$

Finally, it follows via (84, 85, 86, 87) and the definitions that

$$\left(\begin{array}{c}\tau^{\nu+1}\\ \Psi_x\end{array}\right)^{q^\nu}=\left\{\begin{array}{ll}\prod\limits_{\substack{a\in V\\ \xi_1(a)=1}}a\Big/\prod\limits_{\substack{a\in V\\ \xi(a)=1}}a&\text{if }\Theta(\tau^{\nu+1},\Psi_x)=0,\\ \\ (\langle x\rangle+\lfloor T^{-\nu-1}x\rfloor)^{q^\nu}D_\nu&\text{if }\Theta(\tau^{\nu+1},\Psi_x)=1,\end{array}\right.\tag{88}$$

where

$$D_\nu=-\prod_{\substack{a\in V\\ \xi_1(a)=-1}}a=\prod_{\substack{a\in V\\ \xi_1(a)=1}}a=\prod_{i=0}^{\nu-1}(T^{q^\nu}-T^{q^i}),$$

cf. equation (6). The calculation is complete.

10.3 The special point P

We study a certain $\hat{k}_0$-valued point of the group-scheme $\mathcal{H}$ introduced in §4.

10.3.1 Given $u\in\mathcal{U}$, the set $k\cap(u+f\tau\mathcal{O})$ consists of a single element $a\in\mathbb{F}_q[T]$ such that $\deg a<\deg f$. That noted, it is clear that there is a unique group isomorphism

$$(u\mapsto[u]):\mathcal{U}\xrightarrow{\sim}\mathcal{H}(\mathbb{F}_q)$$

such that

$$[u]_\infty(t)=u_\infty(t)\quad(u_\infty=u_\infty(T)\in 1+(1/T)\mathbb{F}_q[[1/T]]\subset\mathcal{O}_\infty^\times)$$

and

$$[u]_f(t)=a(t)\quad(a=a(T)\in k\cap(u+f\tau\mathcal{O}))$$

for every monic $f=f(T)\in\mathbb{F}_q[T]$. Let k^{sep}/k be a separable algebraic closure of k of which k^{ab}/k is a subextension. Let Frob : $\mathcal{H}\to\mathcal{H}$ be the q^{th} power Frobenius endomorphism. According to geometric class field theory (see [Serre]) the set of solutions $X\in\mathcal{H}(k^{\mathrm{sep}})$ of the equation

$$\mathrm{Frob}\,X=[t-T]X\tag{89}$$

forms an $\mathcal{H}(\mathbb{F}_q)$-torsor, and for any solution X we have an explicit reciprocity law

$$X^{\rho(uz\tau^{-N})}=[u]X\quad(u\in\mathcal{U},\ z\in k^\times,\ N\in\mathbb{Z}).\tag{90}$$

In particular, we always have $X\in\mathcal{H}(k^{\mathrm{ab}})$.

10.3.2 Put

$$\mathbf{P} = \prod_{i=0}^{\infty}\left([t - T^{q^i} \otimes 1]^{-1}[t - 1 \otimes T^{q^i}]\right) \in \mathcal{H}(\hat{\mathbf{k}}_0).$$

The product is convergent because

$$[t - T^{q^i} \otimes 1]^{-1}[t - 1 \otimes T^{q^i}] \equiv 1 \bmod \varpi^{q^i}$$

for all i. Since the Lang isogeny $(\mathrm{Frob} - 1) : \mathcal{H} \to \mathcal{H}$ is etale, the functional equation and congruence

$$\mathrm{Frob}\,\mathbf{P} = [t - T \otimes 1][t - 1 \otimes T]^{-1}\mathbf{P}, \quad \mathbf{P} \equiv 1 \bmod \varpi \tag{91}$$

characterize $\mathbf{P}$ uniquely in $\mathcal{H}(\hat{\mathbf{k}}_0)$.

10.3.3 Let $X \in \mathcal{H}(k^{\mathrm{ab}})$ be any solution of (89), let

$$X \otimes 1 \in \mathcal{H}(k^{\mathrm{ab}} \otimes_{\mathbb{F}_q^{\mathrm{ab}}} k^{\mathrm{ab}})$$

be the image of X under the map induced by the ring homomorphism

$$(x \mapsto x \otimes 1) : k^{\mathrm{ab}} \to k^{\mathrm{ab}} \otimes_{\mathbb{F}_q^{\mathrm{ab}}} k^{\mathrm{ab}},$$

let

$$1 \otimes X \in \mathcal{H}(k^{\mathrm{ab}} \otimes_{\mathbb{F}_q^{\mathrm{ab}}} k^{\mathrm{ab}})$$

be defined analogously, and put

$$X \otimes X^{-1} = (X \otimes 1)(1 \otimes X)^{-1}.$$

Then, so we claim, we have

$$\mathbf{P} = X \otimes X^{-1} \in \mathcal{H}(\mathbf{K}_0) \subset \mathcal{H}(\hat{\mathbf{k}}_0). \tag{92}$$

The equality holds because (i) $X \otimes X^{-1}$ satisfies the conditions (91) uniquely characterizing $\mathbf{P}$, and (ii) $X \otimes X^{-1}$ is $\delta\,\mathrm{Gal}(k^{\mathrm{ab}}/k)$-invariant by the explicit reciprocity law (90). The claim is proved.

10.3.4 By combining the observations of the preceding three paragraphs, for any $a \in \mathbb{A}^\times$, we can calculate the image $\mathbf{P}^{(a)}$ of $\mathbf{P}$ under the map $\mathcal{H}(\mathbf{K}_0) \to \mathcal{H}(\mathbf{K})$ induced by the ring homomorphism $(\varphi \mapsto \varphi^{(a)}) : \mathbf{K}_0 \to \mathbf{K}$. We have

$$\mathbf{P}^{(uz\tau^{-N-1})} = [u]\mathbf{P}\begin{cases} \displaystyle\prod_{i=0}^{N}[t - 1 \otimes T^{q^i}]^{-1} & \text{if } N \geqslant 0, \\ \displaystyle\prod_{i=1}^{\nu}[t - 1 \otimes T^{q^{-i}}] & \text{if } N < 0, \end{cases} \tag{93}$$

for all $u \in \mathcal{U}$, $z \in k^\times$ and $N \in \mathbb{Z}$, where $N = -\nu - 1$.

Lemma 10.4. *For all $x = x(T) \in k^\times$ and $a \in \mathbb{A}^\times$ we have*

$$(\theta_{x(t)}(\mathbf{P}))^{(\tau^{-1})} \sim_a \mathcal{C}[\Psi_x]. \tag{94}$$

Recall that θ was defined in §4.1.5 and $\sim$ in §9.6.2.

Proof. Write

$$a = uz\tau^{-N} \in \mathbb{A}^\times \quad (u \in \mathcal{U},\ z \in k^\times,\ N \in \mathbb{Z})$$

in the unique possible way. Without loss of generality we may assume that $z = 1$. Fix monic $f = f(T) \in \mathbb{F}_q[T]$ such that $fx \in \mathbb{F}_q[T]$ and let $\tilde{x} = \tilde{x}(T) \in k$ be uniquely characterized by the

relations

$$[u]_f(t)\langle x(t)\rangle \equiv \langle \tilde{x}(t)\rangle \bmod \mathbb{F}_q[t],$$
$$[u]_\infty(t)\lfloor x(t)\rfloor \equiv \lfloor \tilde{x}(t)\rfloor \bmod (1/t)\mathbb{F}_q[[1/t]],$$

in which case (equivalently)

$$u(x+\tau\mathcal{O}) = \tilde{x}+\tau\mathcal{O}.$$

Let $\psi_x \in \mathbf{K}_0$ be defined by the expression on the left side of (94). Then we have

$$\psi_x^{(a)} = \theta_{x(t)}(\mathbf{P}^{(a\tau^{-1})}) = \theta_{x(t)}([u]\mathbf{P}^{(\tau^{-N-1})}) = \theta_{\tilde{x}(t)}(\mathbf{P}^{(\tau^{-N-1})}) = \psi_{\tilde{x}}^{(\tau^{-N})}$$

and

$$\Psi_x^{(u\tau^{-N})} = \Psi_{\tilde{x}}^{(\tau^{-N})}.$$

After replacing x by $\tilde{x}$, we may simply assume that $a = \tau^{-N}$. All the preceding reductions taken into account, it will be enough to prove that

$$\mathrm{ord}_\Delta\, \psi_x^{(\tau^{-N})} = \Theta(\tau^{-N},\Psi_x), \quad \mathrm{lead}_\Delta\, \psi_x^{(\tau^{-N})} = \left(\begin{array}{c}\tau^{-N}\\ \Psi_x\end{array}\right). \tag{95}$$

If $N \geqslant 0$, then we have via (91) and (93) that

$$\psi_x^{(\tau^{-N})} = \theta_{x(t)}\left(\mathbf{P}^{(\tau^{-N-1})}\right) \equiv \theta_{x(t)}\left(\prod_{i=0}^{N}[t-1\otimes T^{q^i}]^{-1}\right) \bmod \hat{\Delta}_0,$$

and this noted, (95) follows by comparing Proposition 4.2 to formulas (80) and (81). If $N=-\nu-1<0$, then we have via (91) and (93), along with the definition of $\mathbf{P}$, that

$$\begin{aligned}\left(\psi_x^{(\tau^{-N})}\right)^{q^\nu} &= \theta_{x(t)}(\mathrm{Frob}^\nu(\mathbf{P}^{(\tau^{-N-1})}))\\ &= \theta_{x(t)}\left((\mathrm{Frob}^\nu\,\mathbf{P})\textstyle\prod_{i=0}^{\nu-1}[t-1\otimes T^{q^i}]\right)\\ &\equiv \theta_{x(t)}\left([t-T^{q^\nu}\otimes 1]^{-1}\textstyle\prod_{i=0}^{\nu}[t-1\otimes T^{q^i}]\right) \bmod \hat{\Delta}_0^{q^\nu+1},\end{aligned}$$

and this noted, (95) follows by comparing Propositions 4.3 and 4.4 to formulas (86) and (88). □

Lemma 10.5. *Fix $x \in k^\times$ and an integer $n \geqslant 0$. Put*

$$\Phi = \mathbf{1}_{x\tau^{-n+1}\mathcal{O}} - \mathbf{1}_{\tau\mathcal{O}} - (q^n-1)\mathbf{1}_{1+\tau\mathcal{O}}.$$

Then there exists a Coleman unit $\varphi \in \mathbf{k}_0$ such that

$$\mathrm{lead}_\Delta\, \varphi^{(\tau^{-N})} = \left(\begin{array}{c}\tau^{-N}\\ \Phi\end{array}\right)$$

for all $N \gg 0$.

Proof. By example (79) we may assume without loss of generality that $x=1$. Now write $\Phi=\Phi_n$ to keep track of dependence on n. We have

$$\Phi_0 = 0, \quad \Phi_1 = \mathbf{1}_\mathcal{O} - \mathbf{1}_{\tau\mathcal{O}} - (q-1)\mathbf{1}_{1+\tau\mathcal{O}} = \sum_{c\in\mathbb{F}_q^\times}(\mathbf{1}_{1+\tau\mathcal{O}}^{(c)} - \mathbf{1}_{1+\tau\mathcal{O}}).$$

So the lemma holds trivially for $n=0$, and holds for $n=1$ by another application of (79). For all $n \geqslant 0$ we have

$$\Phi_{n+2} - \Phi_{n+1} - q(\Phi_{n+1}-\Phi_n) = \Upsilon^{(\tau^{-n-1})},$$

where

$$\Upsilon = \mathbf{1}_\mathcal{O} - \mathbf{1}_{\tau\mathcal{O}} - q(\mathbf{1}_{\tau\mathcal{O}} - \mathbf{1}_{\tau^2\mathcal{O}}).$$

By induction on n (twice) it suffices to find a Coleman unit $\upsilon \in \mathbf{k}_0$ such that

$$\mathrm{lead}_\Delta\, \upsilon^{(\tau^{-N})} = \left(\begin{array}{c}\tau^{-N}\\ \Upsilon\end{array}\right)$$

$$= \left(\frac{\mathrm{Moore}(T^N,\dots,1)}{\mathrm{Moore}(T^{N-1},\dots,1)}\Big/\frac{\mathrm{Moore}(T^{N-1},\dots,1)^q}{\mathrm{Moore}(T^{N-2},\dots,1)^q}\right)^{q-1} = (T^{q^N}-T)^{q-1}$$

for $N \gg 0$. Clearly,

$$\upsilon = (1\otimes T - T\otimes 1)^{q-1} = \varpi^{q-1}$$

has the desired properties. $\square$

10.6 End of the proof

Fix $x \in k^\times$. By Lemma 10.4 it remains only to prove that $\psi_x = (\theta_{x(t)}(P))^{(\tau^{-1})}$ is a Coleman unit. Fix monic $f \in \mathbb{F}_q[T]$ and an integer $n \geqslant \deg f$. Note that $f^{-1}\tau^{-n+1}\mathcal{O} \supset \tau\mathcal{O}$ and hence $f^{-1}\tau^{-n+1}\mathcal{O}$ is a finite union of cosets of $\tau\mathcal{O}$. Consider the product

$$\varphi = \prod_{\xi \in k^\times \cap f^{-1}\tau^{-n+1}\mathcal{O}} \psi_\xi.$$

Since all the factors in the product by construction are nonzero elements of $\mathbf{K}$, and for suitable f and n the function ψ_x is found among the factors on the right, it suffices simply to show that φ is a Coleman unit. Consider the function

$$\Phi = \sum_{\xi \in k^\times \cap f^{-1}\tau^{-n+1}\mathcal{O}} \Psi_\xi = \mathbf{1}_{f^{-1}\tau^{-n+1}\mathcal{O}} - \mathbf{1}_{\tau\mathcal{O}} - (q^n-1)\mathbf{1}_{1+\tau\mathcal{O}}.$$

By Lemma 10.4 we have $\varphi \sim_a \mathcal{C}[\Phi]$ for all $a \in \mathbb{A}^\times$. It follows by Lemma 10.5 and the uniqueness principle (74) that φ is a Coleman unit. The proof of Conjecture 9.5 in the case $g = 0$ is complete. $\square$

10.7 Remarks

10.7.1 The *solitons* defined in [An92, Thm. 2], and the *Coleman functions* applied in [ABP04], [Si97a] and [Si97b] are all characterized by “interpolation formulas” with righthand sides that can be put into the form of the righthand side of equation (81) above. A similar remark applies to the formula (2) stated in the introduction. For example, by substituting $1 + a/f$ for x in the righthand side of (81), one recovers the righthand side of the interpolation formula stated in [An92, Thm. 2]. For another example, by substituting $x = \alpha/T + T + \beta$ in the righthand side of (81), one recovers the righthand side of formula (2). It follows via the uniqueness principle (74) that solitons and Coleman functions belong to the class of two-variable algebraic functions engendered by Conjecture 9.5 in the genus zero case.

10.7.2 What about the higher genus case? We have four general methods at our disposal for attacking the problem, namely the methods of “rational Fourier analysis” discussed in this paper, along with the methods discussed in the author’s papers [An94], [An96], and [An04]. We are currently working with all these methods in an effort to find a proof of the conjecture.

11. Horizontal specialization of Coleman units

We return to our studies at the full level of generality of Conjecture 9.5. We study embeddings of certain subrings of $\mathbf{K}$ into certain power series rings. Then we draw some general conclusions concerning the operation of “setting the second variable equal to a constant” in a Coleman unit, expressing the results in the form of an easy-to-use calculus. Only at the very end of this section

do we draw any conclusions conditional on Conjecture 9.5.

11.1 Power series constructions

Fix a field $\mathbb{F}$.

11.1.1 Let t, t_1 and t_2 be independent variables. Let $\mathbb{F}((t))$ be the field obtained from the one-variable power series ring $\mathbb{F}[[t]]$ by inverting t. Let $\mathbb{F}((t_1, t_2))$ be the principal ideal domain obtained from the two-variable power series ring $\mathbb{F}[[t_1, t_2]]$ by inverting t_1 and t_2. Given $F = F(t_1, t_2) \in \mathbb{F}((t_1, t_2))$, put

$$F^* = F^*(t_1, t_2) = F(t_2, t_1) \in \mathbb{F}((t_1, t_2)),$$

and let the involutive automorphism $F \mapsto F^*$ of $\mathbb{F}((t_1, t_2))$ be extended in the unique possible way to the fraction field of $\mathbb{F}((t_1, t_2))$.

11.1.2 Let $0 \neq F = F(t_1, t_2) \in \mathbb{F}((t_1, t_2))$ be given. There is a unique factorization

$$F(t_1, t_2) = t_1^{\ell_1} t_2^{\ell_2} G(t_1, t_2)$$

where $\ell_1, \ell_2 \in \mathbb{Z}$ and $G(t_1, t_2) \in \mathbb{F}[[t_1, t_2]]$ is divisible by neither t_1 nor t_2. In turn, there is by the Weierstrass preparation theorem a unique factorization

$$G(t_1, t_2) = U(t_1, t_2)(t_1^m + t_2 H(t_1, t_2))$$

where $U(t_1, t_2) \in \mathbb{F}[[t_1, t_2]]^\times$, m is a nonnegative integer and $H(t_1, t_2) \in \mathbb{F}[[t_1, t_2]]$ is a polynomial in t_1 of degree $< m$. We now define

$$\mathrm{wt}\, F = \ell_2 \in \mathbb{Z}, \quad \epsilon[F] = \epsilon[F](t) = t^{\ell_1 + m} U(t, 0) \in \mathbb{F}((t))^\times.$$

Also put $\mathrm{wt}\, 0 = +\infty$. We have

$$t^{-\mathrm{wt}\, F^*} \epsilon[F] \in \mathbb{F}[[t]]^\times \Leftrightarrow m = 0 \Leftrightarrow F \in \mathbb{F}((t_1, t_2))^\times. \tag{96}$$

The function wt is an additive valuation of $\mathbb{F}((t_1, t_2))$. We extend wt in the unique possible way to a normalized additive valuation of the fraction field of $\mathbb{F}((t_1, t_2))$. The $\mathbb{F}((t))^\times$-valued function ϵ defined on $\mathbb{F}((t_1, t_2)) \setminus \{0\}$ is multiplicative. We extend ϵ in the unique possible way to a homomorphism from the multiplicative group of the fraction field of $\mathbb{F}((t_1, t_2))$ to $\mathbb{F}((t))^\times$.

11.2 Embeddings, valuations, and leading coefficients

Fix a finite subextension K/k of $k^{\mathrm{ab}}_{\mathrm{perf}}/k$. Let $\mathbb{F}$ be the field of constants of K. Put $K^{[2]} = K \otimes_{\mathbb{F}} K$, which is a Dedekind domain.

11.2.1 Given a place w of K with residue field equal to $\mathbb{F}$, a uniformizer π in the completion K_w, and $x \in K$, let $x_{K,w,\pi}(t) \in \mathbb{F}((t))$ be the unique Laurent series such that the equality $x_{K,w,\pi}(\pi) = x$ holds in K_w. Put

$$\iota_{K,w,\pi} = (x \otimes y \mapsto x \cdot y_{K,w,\pi}(t)) : K^{[2]} \to K((t)),$$

thus defining an embedding of the ring $K^{[2]}$ into the field $K((t))$. We extend $\iota_{K,w,\pi}$ in the unique possible way to an embedding of the fraction field of $K^{[2]}$ into $K((t))$.

11.2.2 With w and π as above, and given nonzero φ in the fraction field of $K^{[2]}$, write

$$\iota_{K,w,\pi}\varphi = \sum_{i=i_0}^{\infty} e_i t^i \quad (i_0 \in \mathbb{Z}, \quad e_i \in K, \quad e_{i_0} \neq 0), \tag{97}$$

and put

$$\mathrm{wt}_{K,w,\pi}\varphi = i_0, \quad \epsilon_{K,w,\pi}\varphi = e_{i_0}.$$

Also put $\mathrm{wt}_{K,w,\pi}0 = +\infty$. The function $\mathrm{wt}_{K,w,\pi}$ is independent of π. We write $\mathrm{wt}_{K,w}$ hereafter. The function $\mathrm{wt}_{K,w}$ is a normalized additive valuation of the fraction field of $K^{[2]}$. The function $\epsilon_{K,w,\pi}$ is a homomorphism from the multiplicative group of the fraction field of $K^{[2]}$ to $K^\times$. While $\epsilon_{K,w,\pi}$ does depend on π, the dependence is rather mild: given another uniformizer $\pi' \in K$ at w, the ratio $\epsilon_{K,w,\pi}/\epsilon_{K,w,\pi'}$ is a homomorphism from the fraction field of $K^{[2]}$ to $\mathbb{F}^\times$ of the form $F \mapsto c^{\mathrm{wt}_{K,w} F}$ for some $c \in \mathbb{F}^\times$.

11.2.3 Let w, π and φ be as above, but assume now that φ belongs to the fraction field of $K^{[2]} \cap \mathbf{K}$. Fix $a \in \mathbb{A}^\times$ such that $\|a\| \geqslant 1$. Then $\varphi^{(a)}$ is defined and belongs again to the fraction field of $K^{[2]} \cap \mathbf{K}$. We claim that

$$\mathrm{wt}_{K,w}\varphi^{(a)} = \|a\|\mathrm{wt}_{K,w}\varphi, \ \ \epsilon_{K,w,\pi}(\varphi^{(a)}) = (\epsilon_{K,w,\pi}\varphi)^{\rho(a)}. \tag{98}$$

In any case, notation as in (97), we must have

$$\iota_{K,w,\pi}\varphi^{(a)} = \sum_{i=i_0}^{\infty} e_i^{\rho(a)} t^{i\|a\|}$$

on account of the rule $x \otimes y \mapsto x^{\rho(a)} \otimes y^{\|a\|}$ by which $\varphi^{(a)}$ is defined. The claim follows.

11.2.4 Given for $i = 1, 2$ a place w_i of K with residue field $\mathbb{F}$ and a uniformizer π_i in K_{w_i}, put

$$\iota_{K,w_1,w_2,\pi_1,\pi_2} = (x \otimes y \mapsto x_{K,w_1,\pi_1}(t_1)y_{K,w_2,\pi_2}(t_2)) : K^{[2]} \to \mathbb{F}((t_1, t_2)),$$

thus defining an embedding of the ring $K^{[2]}$ into $\mathbb{F}((t_1, t_2))$. We extend $\iota_{K,w_1,w_2,\pi_1,\pi_2}$ in the unique possible way to an embedding of the fraction field of $K^{[2]}$ into the fraction field of $\mathbb{F}((t_1, t_2))$. Clearly, we have

$$\mathrm{wt}_{K,w_2}\varphi = \mathrm{wt}\, \iota_{K,w_1,w_2,\pi_1,\pi_2}\varphi. \tag{99}$$

It is not difficult to verify that the equality

$$\epsilon_{K,w_2,\pi_2}\varphi = \epsilon[\iota_{K,w_1,w_2,\pi_1,\pi_2}\varphi](\pi_1) \tag{100}$$

holds in K_{w_1}. If φ belongs to the fraction field of $\mathbf{K} \cap K^{[2]}$, then $\varphi^\dagger$ is defined and belongs again to the fraction field of $\mathbf{K} \cap K^{[2]}$, and we have

$$(\iota_{K,w_1,w_2,\pi_1,\pi_2}\varphi)^* = \iota_{K,w_2,w_1,\pi_2,\pi_1}(\varphi^\dagger), \tag{101}$$

on account of the rule $x \otimes y \mapsto y \otimes x$ defining the "dagger" operation.

Lemma 11.3. *Let K, $\mathbb{F}$ and $K^{[2]}$ be as in §11.2. For $i = 1, 2$ let w_i be a place of K with residue field equal to $\mathbb{F}$, let π_i be a uniformizer in K_{w_i}, and let v_i be the unique place of k below w_i. Assume that $v_1 \neq v_2$. Fix $a \in \mathbb{A}^\times$ and put*

$$I(a) = \ker\left((x \otimes y \mapsto x^{\rho(a)}y^{\|a\|}) : K^{[2]} \to K_{\mathrm{perf}}\right).$$

Then there exists $\psi \in I(a)$ such that

$$\iota_{K,w_1,w_2,\pi_1,\pi_2}\psi \in \mathbb{F}[[t_1, t_2]]^\times.$$

Proof. For any $x \in K$ we have

$$\delta_a x = (x^{\rho(a)} \otimes 1 - 1 \otimes x^{\|a\|})^{\max(1,\|a\|^{-1})} \in I(a).$$

Since the additive valuations
$$x \mapsto \operatorname{ord}_{w_1} x^{\rho(a)}, \quad x \mapsto \operatorname{ord}_{w_2} x$$
are inequivalent (they already have inequivalent restrictions to k by hypothesis), we can find $x \in K$ by the Artin-Whaples approximation theorem such that
$$\operatorname{ord}_{w_1} x^{\rho(a)} > 0, \quad \operatorname{ord}_{w_2} x = 0.$$
Then $\psi = \delta_a x$ has the desired property. □

PROPOSITION 11.4. *Let K, $\mathbb{F}$ and $K^{[2]}$ be as in §11.2. For $i = 1, 2$ let w_i be a place of K with residue field equal to $\mathbb{F}$, let π_i be a uniformizer in K_{w_i}, and let v_i be the unique place of k below w_i. Assume that $v_1 \neq v_2$. For all Coleman units φ belonging to the fraction field of $K^{[2]}$, we have* $\operatorname{ord}_{w_1} \epsilon_{K,w_2,\pi_2}\varphi = \operatorname{wt}_{K,w_1}\varphi^\dagger$.

Proof. By the preceding lemma and the definition of a Coleman unit, we can find $\psi \in K^{[2]}$ such that
$$\psi\varphi^{\pm 1} \in K^{[2]}, \ \ \iota_{K,w_1,w_2,\pi_1,\pi_2}(\psi) \in \mathbb{F}[[t_1, t_2]]^\times,$$
in which case it follows that
$$\iota_{K,w_1,w_2,\pi_1,\pi_2}(\varphi) \in \mathbb{F}((t_1, t_2))^\times.$$
We then have
$$\begin{aligned} \operatorname{ord}_{w_1} \epsilon_{K,w_2,\pi_2} &= \operatorname{ord}_{w_1} \epsilon[\iota_{K,w_2,\pi_2}\varphi](\pi_1) \\ &= \operatorname{wt}\ (\iota_{K,w_1,w_2,\pi_1,\pi_2}\varphi)^* \\ &= \operatorname{wt} \iota_{K,w_2,w_1,\pi_2,\pi_1}(\varphi^\dagger) \\ &= \operatorname{wt}_{K,w_1,\pi_1}\varphi^\dagger \end{aligned}$$
at the first equality by (100), at the second by (96), at the third by (101), and at the fourth by (99). □

PROPOSITION 11.5. *Let K, $\mathbb{F}$ and $K^{[2]}$ be as in §11.2. Let w be a place of K with residue field equal to $\mathbb{F}$ and let π be a uniformizer in K_w. Let v be the place of k below w and let τ be a uniformizer in k_v. For every nonzero φ belonging to the fraction field of $K^{[2]} \cap \mathbf{K}$, we have*
$$\lim_{n\to\infty} \operatorname{ord}_w \left(\epsilon_{K,w,\pi}\varphi - \pi^{-q_v^{nd}\operatorname{wt}_{K,w}\varphi} \operatorname{lead}_\Delta \varphi^{(i_v^\times(\tau^{-nd}))}\right) = +\infty,$$
where $d = [K : k]$, and ord_w is the normalized additive valuation of K associated to w.

We are indebted to D. Thakur for teaching us about this sort of convergence phenomenon. (See, for example, [Thak, Thm. 4.8.1].)

Proof. In general, φ is a quotient of elements of $K^{[2]} \cap \mathbf{K}$. But since we get the result if we can prove it for numerator and denominator of φ separately, we may assume without loss of generality that $\varphi \in K^{[2]} \cap \mathbf{K}$. Put
$$\iota_{K,w,w,\pi,\pi}(\varphi) = F(t_1, t_2) \in \mathbb{F}((t_1, t_2)).$$
Write
$$F(t_1, t_2) = t_1^{\ell_1} t_2^{\ell_2} G(t_1, t_2),$$
where $G(t_1, t_2) \in \mathbb{F}[[t_1, t_2]]$ is divisible neither by t_1 nor t_2. By the Weierstrass preparation theorem, write
$$G(t_1, t_2) = (t_1^m + t_2 H(t_1, t_2))U(t_1, t_2),$$
where m is a nonnegative integer, $H(t_1, t_2) \in \mathbb{F}[[t_1, t_2]]$ is a polynomial in t_1 of degree $< m$, and $U(t_1, t_2) \in \mathbb{F}[[t_1, t_2]]^\times$. We have
$$\operatorname{wt}_{K,w}\varphi = \ell_2, \quad \epsilon_{K,w,\pi}\varphi = \pi^{\ell_1} U(\pi, 0)$$

by (99), (100) and the definitions. Write

$$\varphi = \sum_i x_i \otimes y_i \quad (x_i, y_i \in K).$$

Note that for $n \geqslant 0$ we have

$$\varphi^{(i_v^\times(\tau^{-nd}))} = \sum_i x_i^{\rho(i_v^\times(\tau^{-n}))^d} \otimes y_i^{q_v^{nd}} = \sum_i x_i \otimes y_i^{q_v^{nd}},$$

also we have $\#\mathbb{F}|q_v^d$, and hence equality

$$\mathrm{lead}_\Delta\, \varphi^{(i_v^\times(\tau^{-nd}))} = \sum_i x_i y_i^{q_v^{nd}} = F(\pi, \pi^{q_v^{nd}})$$

$$= \pi^{\ell_1 + q_v^{nd}\ell_2}(\pi^m + \pi^{q_v^{nd}} H(\pi, \pi^{q_v^{nd}}))U(\pi, \pi^{q_v^{nd}})$$

holds in the completion K_w. The result follows. □

11.6 The functions wt_v and $\epsilon_{\bar{v}}$

We boil the preceding somewhat complicated considerations down to a few easy-to-apply rules.

11.6.1 Let v be a place of k, let $\bar{v}$ be a place of $k_{\mathrm{perf}}^{\mathrm{ab}}$ above v and let $\mathrm{ord}_{\bar{v}}$ be the unique additive valuation of $k_{\mathrm{perf}}^{\mathrm{ab}}$ belonging to the place $\bar{v}$ which extends the normalized additive valuation ord_v of k. For every nonzero φ in the fraction field of $\mathbf{K}$ we define

$$\begin{aligned} \mathrm{wt}_v\varphi &= \mathrm{ord}_{\bar{v}}\,\pi \cdot \mathrm{wt}_{K,w}\varphi, \\ \epsilon_{\bar{v}}\varphi &= \begin{cases} \epsilon_{K,w,\pi}\varphi & \text{if } \mathrm{wt}_v\varphi = 0, \\ \epsilon_{K,w,\pi}\varphi \bmod (\mathbb{F}_q^{\mathrm{ab}})^\times & \text{if } \mathrm{wt}_v\varphi \neq 0, \end{cases} \end{aligned}$$

where:

- K/k is any finite subextension of $k_{\mathrm{perf}}^{\mathrm{ab}}/k$ such that
  - φ belongs to the fraction field of $K^{[2]}$,
  - w is the place of K below $\bar{v}$,
  - π is a uniformizer at w, and
  - the constant field $\mathbb{F}$ of K equals the residue field of w.

We are obliged to check that wt_v and $\epsilon_{\bar{v}}$ are well-defined. In any case, for every φ we can find suitable K as above. Moreover, for fixed φ, the expression on the right side of the definition of $\epsilon_{\bar{v}}$ is easily verified to depend only on $\bar{v}$, and the same is true of the expression on the right side of the definition of $\mathrm{wt}_v\varphi$. Finally, the latter depends only on v by Proposition 11.5. Thus wt_v and $\epsilon_{\bar{v}}$ are indeed well-defined. For convenience put $\mathrm{wt}_v 0 = +\infty$. Then the function wt_v is an additive valuation of the fraction field of $\mathbf{K}$. The function $\epsilon_{\bar{v}}$ is a homomorphism from the multiplicative group of the fraction field of $\mathbf{K}$ to $(k_{\mathrm{perf}}^{\mathrm{ab}})^\times/(\mathbb{F}_q^{\mathrm{ab}})^\times$ which on the subgroup $\{\mathrm{wt}_{\bar{v}} = 0\}$ is refined to a homomorphism to $(k_{\mathrm{perf}}^{\mathrm{ab}})^\times$.

11.6.2 We claim that the following relations hold:

$$\mathrm{wt}_v\varphi^{(a)} = \|a\|\mathrm{wt}_v\varphi, \quad \epsilon_{\bar{v}}\varphi^{(a)} = (\epsilon_{\bar{v}}\varphi)^{\rho(a)} \tag{102}$$

$$\mathrm{ord}_v\,\mathrm{lead}_\Delta\,\varphi^{(i_v^\times(\tau^{-n!}))} = \mathrm{ord}_{\bar{v}}\,\epsilon_{\bar{v}}\varphi + q_v^{n!}\mathrm{wt}_v\varphi \quad (n \gg 0) \tag{103}$$

$$\mathrm{wt}_v\varphi = 0 \Rightarrow \lim_{n\to\infty} \mathrm{ord}_{\bar{v}}\left(\epsilon_{\bar{v}}\varphi - \mathrm{lead}_\Delta\,\varphi^{(i_v^\times(\tau^{-n!}))}\right) = +\infty \tag{104}$$

$$\mathrm{ord}_{\bar{v}_1}\,\epsilon_{\bar{v}_2}\varphi = \mathrm{wt}_{v_1}\varphi^\dagger \quad (v_1 \neq v_2) \tag{105}$$

Here $\bar{v}$ (resp., $\bar{v}_1$, $\bar{v}_2$) are places of $k^{\mathrm{ab}}_{\mathrm{perf}}$ above places v (resp., v_1, v_2) of k, φ is a Coleman unit, $a \in \mathbb{A}^\times$, and $\tau \in k$ is a uniformizer at v. Relation (98) justifies (102). Proposition 11.5 justifies relations (103) and (104). Proposition 11.4 justifies (105).

11.6.3 Keeping the notation of the preceding paragraph, suppose further that $\varphi = \mathcal{C}[\Phi]$ for some $\Phi \in \mathrm{Sch}_{00}(\mathbb{A})$. (Notice that not until now have we invoked Conjecture 9.5.) We claim that

$$\mathrm{wt}_v\varphi = \int \tilde{\Phi}(i_v(t))d\mu_v^\times(t), \tag{106}$$

$$\mathrm{wt}_v\varphi^\dagger = \int \Phi(i_v(t))d\mu_v^\times(t), \tag{107}$$

$$\mathrm{ord}_{\bar{v}}\,\epsilon_{\bar{v}}\varphi = \int \Phi(i_v(t))d\mu_v^\times(t) + \int \Theta(i_v^\times(t),\Phi)d\mu_v^\times(t). \tag{108}$$

Equations (106) and (108) are proved by comparing (103) to the simplified version (69) of the adelic Stirling formula. Equation (107) follows from (106) since, as remarked in §9.6, we have $\mathcal{C}[\tilde{\Phi}] = \mathcal{C}[\Phi]^\dagger$.

12. A conditional recipe for the Stark unit

We continue in the setting of Conjecture 9.5. Let K/k be a finite subextension of k^{ab}/k. Let S be a finite set of places of k. Assume the following:

- Some place $\infty \in S$ splits completely in K.
- The set $S_0 = S \setminus \{\infty\}$ is nonempty.
- All places of k ramified in K belong to S_0.

We are going to show that Tate's formulation $\mathrm{St}(K/k, S)$ [Tate, p. 89, Conj. 2.2] of the Stark conjecture is a consequence of Conjecture 9.5. After introducing suitable notation and making a convenient reduction, we recall the statement of $\mathrm{St}(K/k, S)$ in detail below. We must of course remark that since we are working in the function field situation, $\mathrm{St}(K/k, S)$ is already a theorem due to Deligne [Tate] and (independently) to Hayes [Ha85]. The point of deriving $\mathrm{St}(K/k, S)$ from our conjecture is to establish that the latter does in fact refine the former.

12.1 Notation and a reduction

12.1.1 For each place v of k we fix the following objects:

- Let $\tau_v \in \mathbb{A}^\times$ be the image of a fixed choice of uniformizer of k_v under the map $i_v^\times : k_v^\times \to \mathbb{A}^\times$.
- Put $\langle\tau_v\rangle = \{\tau_v^n | n \in \mathbb{Z}\} \subset \mathbb{A}^\times$.
- Fix a place $\bar{v}$ of $k^{\mathrm{ab}}_{\mathrm{perf}}$ above v.
- Let $\mathrm{ord}_{\bar{v}}$ be the unique additive valuation of $k^{\mathrm{ab}}_{\mathrm{perf}}$ belonging to $\bar{v}$ and extending the normalized additive valuation ord_v of k.

12.1.2 According to Tate [Tate, Prop. 3.5, p. 92], $\mathrm{St}(K/k, S)$ implies $\mathrm{St}(K'/k, S)$ for any subextension K'/k of K/k. Accordingly, after choosing a suitable finite subextension $\tilde{K}/K$ of k^{ab}/K and replacing $(K/k, S)$ by $(\tilde{K}/k, S)$, we may assume without loss of generality that the data $(K/k, S)$ satisfy the following further condition:

- $\ker \rho_{K/k} = k^\times \mathcal{U}\langle\tau_\infty\rangle \subset \mathbb{A}^\times$, where $\mathcal{U} \subset \mathcal{O}^\times$ is an open subgroup with the following properties:
  - $\mathcal{U} \supset i_v^\times(\mathcal{O}_v^\times)$ for all places v of k not belonging to S_0.

$*$ $\mathcal{U} \cap k^\times = \{1\}$.

Note that under this further condition the constant field $\mathbb{F}$ of K has cardinality q_∞, and hence there are exactly $q_\infty - 1$ roots of unity in K.

12.1.3 For $T = S_0$ or $T = S$, consider the Euler product

$$\theta_T(s) = \prod_{v \notin T} (1 - F_v q_v^{-s})^{-1} = \sum_{\sigma \in G} \zeta_T(s,\sigma)\sigma^{-1} \in \mathbb{C}[G] \quad (\Re(s) > 1)$$

extended over places v of k not in T, where $F_v = \rho_{K/k}(\tau_v) \in G$ is the geometric Frobenius at v (cf. [Tate, p. 86, Prop. 1.6]). If $T = S_0$, we drop the subscript and write simply $\theta(s)$ and $\zeta(s,\sigma)$. It is well-known that $\theta_T(s)$ continues meromorphically to the entire s-plane with no singularity other than a pole at $s = 1$. Note that we have

$$\theta_S(0) = 0, \quad \theta'_S(0) = \log q_\infty \cdot \theta(0)$$

(cf. [Tate, p. 86, Cor. 1.7]) since ∞ splits completely in K/k.

12.1.4 Let $U \subset K^\times$ be the G-submodule consisting of x satisfying the following condition:

– $\mathrm{ord}_{\bar{v}}\, x^\sigma = \begin{cases} \mathrm{ord}_{\bar{v}}\, x & \text{if } S_0 = \{v\}, \\ 0 & \text{otherwise,} \end{cases}$ for all places v of k distinct from ∞ and $\sigma \in G$.

Let $U^{\mathrm{ab}} \subset U$ be the G-submodule consisting of x satisfying the following further condition:

– $K(x^{1/(q_\infty - 1)})/k$ is an abelian extension.

It is convenient to define a group homomorphism

$$\mathbf{ord} = \left(x \mapsto \sum_{\sigma \in G} (\mathrm{ord}_{\overline{\infty}}\, x^\sigma)\sigma^{-1} \right) : U \to \mathbb{C}[G].$$

Note that $\mathbf{ord}$ is G-equivariant and $\ker \mathbf{ord} = \mathbb{F}^\times$.

12.1.5 The *Stark unit*

$$\epsilon(K/k, S) \in U^{\mathrm{ab}}$$

predicted to exist by conjecture

$$\mathrm{St}(K/k, S) \quad ([\text{Tate, p. 89, Conj. 2.2}]),$$

given the choice $\overline{\infty}|_K$ of a place of K above ∞, is uniquely determined up to a factor in $\mathbb{F}^\times$ by the formulas

$$(q_\infty - 1)\zeta(0,\sigma) = \mathrm{ord}_{\overline{\infty}}\, \epsilon(K/k,S)^\sigma \text{ for all } \sigma \in G,$$

or, equivalently, the formula

$$(q_\infty - 1)\theta(0) = \mathbf{ord}\, \epsilon(K/k, S). \tag{109}$$

The rest of our work in §12 is devoted to working out a “recipe” for the Stark unit in terms of the transformation $\mathcal{C}$ defined by Conjecture 9.5.

12.2 “Ingredient list”

12.2.1 Let $\overline{S}$ be the set of places of k not belonging to S. Let Γ be the subgroup of $\mathbb{A}^\times$ generated by $\{\tau_v | v \in \overline{S}\}$. Let $\mathbb{Z}[\Gamma]$ be the group ring of Γ over the integers. Let $\mathcal{J}$ be the kernel of the ring homomorphism $\mathbb{Z}[\Gamma] \to \mathbb{Z}[1/q]$ induced by the restriction of the idele norm function $\|\cdot\|$ to Γ. Since Γ is a free abelian group with basis $\{\tau_v | v \in \overline{S}\}$, the ring $\mathbb{Z}[\Gamma]$ may be viewed as the ring of

Laurent polynomials with integral coefficients in independent variables τ_v indexed by $v \in \overline{S}$. From the latter point of view it is clear that the set

$$\{1 - q_v \cdot \tau_v | v \in \overline{S}\} \subset \mathbb{Z}[\Gamma]$$

generates $\mathcal{J}$ as an ideal of $\mathbb{Z}[\Gamma]$. Here, and in similar contexts below, the expression $1 - q_v \cdot \tau_v$ is to be viewed as a formal $\mathbb{Z}$-linear combination of elements of Γ.

12.2.2 Put

$$\mathcal{U}(\infty) = \{a = [a_v] \in \mathcal{U} | \|a_\infty - 1\|_\infty < 1\}.$$

Let $K(\infty)/k$ be the unique subextension of k^{ab}/k such that

$$\ker \rho_{K(\infty)/k} = k^\times \mathcal{U}(\infty)\langle\tau_\infty\rangle$$

and put

$$G(\infty) = \mathrm{Gal}(K(\infty)/k).$$

Crucially, the constant field $\mathbb{F}$ of K is also the constant field of $K(\infty)$.

12.2.3 Put

$$\begin{array}{rcl}\Pi & = & \Gamma \cap k^\times \mathcal{U}\langle\tau_\infty\rangle, \\ \Pi(\infty) & = & \Gamma \cap k^\times \mathcal{U}(\infty)\langle\tau_\infty\rangle, \\ \Pi_1(\infty) & = & \Gamma \cap k^\times \mathcal{U}(\infty) = \{a \in \Pi(\infty) |\ \|a\| = 1\}.\end{array}$$

Extend the group homomorphism $\Gamma \xrightarrow{\rho_{K/k}} G$ to a ring homomorphism $\mathbb{Z}[\Gamma] \xrightarrow{\rho_{K/k}} \mathbb{Z}[G]$, and define $\mathbb{Z}[\Gamma] \xrightarrow{\rho_{K(\infty)/k}} \mathbb{Z}[G(\infty)]$ analogously. Let $J(\infty) \subset \mathbb{Z}[G(\infty)]$ be the annihilator of $\mathbb{F}^\times$. For any subgroup $\Gamma' \subset \Gamma$ let $\mathcal{I}(\Gamma') \subset \mathbb{Z}[\Gamma]$ be the ideal generated by differences of elements of Γ'. From the exactness of the sequence

$$1 \to \Pi \to \Gamma \xrightarrow{\rho_{K/k}} G \to 1,$$

which is well-known, we deduce an exact sequence

$$0 \to \mathcal{I}(\Pi) \subset \mathbb{Z}[\Gamma] \xrightarrow{\rho_{K/k}} \mathbb{Z}[G] \to 0. \tag{110}$$

Analogously we have an exact sequence

$$0 \to \mathcal{I}(\Pi(\infty)) \subset \mathbb{Z}[\Gamma] \xrightarrow{\rho_{K(\infty)/k}} \mathbb{Z}[G(\infty)] \to 0. \tag{111}$$

We claim that we have an exact sequence

$$0 \to \mathcal{I}(\Pi_1(\infty)) + \mathcal{I}(\Pi(\infty)) \cdot \mathcal{J} \subset \mathcal{J} \xrightarrow{\rho_{K(\infty)/k}} J(\infty) \to 0. \tag{112}$$

Exactness at $\mathcal{J}$ follows from Lemma 5.1. The set

$$\{v \in \overline{S} | 1 - q_v \rho_{K(\infty)/k}(\tau_v)\}$$

by [Tate, p. 82, Lemme 1.1] generates $J(\infty)$ as an ideal of $\mathbb{Z}[G(\infty)]$, whence exactness at $J(\infty)$. The claim is proved.

12.2.4 From exactness of (111) and (112) it follows that there exists $\mathbf{a}_\infty \in \mathcal{J}$ such that $\rho_{K(\infty)/k}(\mathbf{a}_\infty) = q_\infty - 1$.

12.2.5 Put

$$\mathcal{V} = \left\{ a = [a_v] \in \mathcal{O} \,\middle|\, v \in S_0 \Rightarrow a_v \neq 0, \quad \prod_{v \in S_0} i_v^\times(a_v) \in \mathcal{U} \right\} \subset \mathcal{O},$$

which is an open compact subset of $\mathbb{A}$. Note that $\mathcal{V}$ is stable under the action of the group $\mathcal{U}$ by multiplication. Extend the map

$$(a \mapsto \mathbf{1}_{\mathcal{V}}^{(a)}) : \Gamma \to \mathrm{Sch}_0(\mathbb{A})$$

$\mathbb{Z}$-linearly to a homomorphism

$$(\mathbf{a} \mapsto \mathbf{1}_{\mathcal{V}}^{(\mathbf{a})}) : \mathbb{Z}[\Gamma] \to \mathrm{Sch}_0(\mathbb{A})$$

of abelian groups, noting that

$$\mathbf{1}_{\mathcal{V}}^{(a\mathbf{a})} = (\mathbf{1}_{\mathcal{V}}^{(\mathbf{a})})^{(a)}, \quad \mathbf{a} \in \mathcal{J} \Leftrightarrow \mathbf{1}_{\mathcal{V}}^{(\mathbf{a})} \in \mathrm{Sch}_{00}(\mathbb{A}) \tag{113}$$

for all $\mathbf{a} \in \mathbb{Z}[\Gamma]$ and $a \in \Gamma$.

12.2.6 We claim that for all $\mathbf{a} \in \mathcal{J}$ we have

$$\int \mathcal{F}_0[\mathbf{1}_{\mathcal{V}}^{(\mathbf{a})}](i_\infty(t))d\mu_\infty^\times(t) = 0. \tag{114}$$

In any case, the set of $\mathbf{a} \in \mathcal{J}$ satisfying the equation above forms an ideal of $\mathbb{Z}[\Gamma]$, and so to prove the claim we may assume without loss of generality that $\mathbf{a} = 1 - q_v \cdot \tau_v$ for some $v \in \overline{S}$, in which case already by the scaling rule (60) the integrand above vanishes identically as a function of $t \in k_\infty^\times$. The claim is proved.

12.2.7 So far all our constructions and definitions make sense unconditionally. But from this point onward we must assume that Conjecture 9.5 holds so that the transformation $\mathcal{C}$ is defined. From (106) and (114) it follows that for every $\mathbf{a} \in \mathcal{J}$ we have

$$\mathrm{wt}_\infty \mathcal{C}[\mathbf{1}_{\mathcal{V}}^{(\mathbf{a})}] = 0$$

and hence

$$\epsilon(\mathbf{a}) = \epsilon_{\overline{\infty}} \mathcal{C}[\mathbf{1}_{\mathcal{V}}^{(\mathbf{a})}] \in (k_{\mathrm{perf}}^{\mathrm{ab}})^\times$$

is well-defined. Note that by (102) and (113) the homomorphism

$$\epsilon : \mathcal{J} \to (k_{\mathrm{perf}}^{\mathrm{ab}})^\times$$

is Γ-equivariant, where we view the target in the natural way as a Γ-module via the reciprocity law ρ.

Theorem 12.3 “Recipe”. *Hypotheses and notation as above, we have $\epsilon(K/k, S) = \epsilon(\mathbf{a}_\infty)$.*

We stress that this result is conditional on Conjecture 9.5. The proof consists of an analysis of the Γ-equivariant homomorphism ϵ proceeding by way of several lemmas.

Lemma 12.4. $\epsilon(\mathcal{J}) \subset \left(\bar{\infty}\text{-*closure of* } k \text{ *in* } k_{\mathrm{perf}}^{\mathrm{ab}}\right)^\times \subset (k^{\mathrm{ab}})^\times$.

Proof. For all $\mathbf{a} \in \mathcal{J}$ we have

$$\lim_{n \to \infty} \mathrm{ord}_{\bar{\infty}} \left(\epsilon(\mathbf{a}) - \left(\begin{array}{c} \tau_\infty^{-n!} \\ \\ \mathbf{1}_{\mathcal{V}}^{(\mathbf{a})} \end{array} \right) \right) = +\infty$$

by limit formula (104) and the definitions. Moreover, the Catalan symbol in question takes values in $k^\times$ for all $n \gg 0$ by Proposition 8.3. The result follows. □

Lemma 12.5. $\ker \epsilon \supset \mathcal{I}(\Pi) \cdot \mathcal{J} + \mathcal{I}(\Pi_1(\infty))$.

Proof. Fix $a \in \Pi$ arbitrarily and write $a = ux\tau_\infty^N$ with $u \in \mathcal{U}$, $x \in k^\times$ and $N \in \mathbb{Z}$, noting that since $\mathcal{U} \cap k^\times = \{1\}$, this factorization is unique. For all $\mathbf{a} \in \mathcal{J}$ we have

$$\begin{aligned} \epsilon(\mathbf{a}a) = \epsilon_{\overline{\infty}}\mathcal{C}[\mathbf{1}_{\mathcal{V}}^{(\mathbf{a}a)}] &= \epsilon_{\overline{\infty}}\mathcal{C}[\mathbf{1}_{\mathcal{V}}^{(\mathbf{a}x\tau_\infty^n)}] \\ &= \epsilon_{\overline{\infty}}\mathcal{C}[(\mathbf{1}_{\mathcal{V}}^{(\mathbf{a})})^{(x\tau_\infty^n)}] \\ &= \epsilon_{\overline{\infty}}(\mathcal{C}[\mathbf{1}_{\mathcal{V}}^{(\mathbf{a})}]^{(\tau_\infty^n)}) = \epsilon(\mathbf{a})^{\rho(\tau_\infty^n)} = \epsilon(\mathbf{a}) \end{aligned}$$

where the fourth equality is justified by the $\mathbb{A}^\times$-equivariance of $\mathcal{C}$ noted in §9.6, the fifth by (102), and the last by the preceding lemma. Therefore we have $\ker\epsilon \supset \mathcal{I}(\Pi)\cdot\mathcal{J}$. Now suppose that $a \in \Pi_1(\infty)$. Then necessarily $N = 0$ and $xu_\infty = 1$, hence $\|x-1\|_\infty < 1$, and hence

$$\epsilon(\mathbf{1}_{\mathcal{V}}^{(a)} - \mathbf{1}_{\mathcal{V}}) = \epsilon_{\overline{\infty}}\mathcal{C}[\mathbf{1}_{\mathcal{V}}^{(x)} - \mathbf{1}_{\mathcal{V}}] = \epsilon_{\overline{\infty}}(1 \otimes x^{\mu(\mathcal{V})}) = 1,$$

where the middle equality is justified by example (79). Therefore we have $\ker\epsilon \supset \mathcal{I}(\Pi_1(\infty))$. □

LEMMA 12.6. $\epsilon(\mathcal{J}) \subset U^{\mathrm{ab}}$.

Proof. By exactness of the sequence (110) and the previous two lemmas, ϵ takes values in $K^\times$. We claim that

$$\int \mathbf{1}_{\mathcal{V}}^{(\mathbf{a}a)}(i_v(t))d\mu_v^\times(t) = \begin{cases} \int \mathbf{1}_{\mathcal{V}}^{(\mathbf{a})}(i_v(t))d\mu_v^\times(t) & \text{if } S_0 = \{v\} \\ 0 & \text{if } S_0 \neq \{v\} \end{cases} \tag{115}$$

for all $\mathbf{a} \in \mathcal{J}$, $a \in \Gamma$ and places v of k. If $S_0 \neq \{v\}$, then we have $b\mathcal{V} \cap i_v(k_v) = \emptyset$ for all $b \in \mathbb{A}^\times$ and *a fortiori* (115) holds; otherwise, if $S_0 = \{v\}$, we get (115) by an evident manipulation of integrals. The claim is proved. It follows by formula (107) of §11.6 that

$$\mathrm{wt}_v\mathcal{C}[\mathbf{1}_{\mathcal{V}}^{(\mathbf{a}a)}]^\dagger = \begin{cases} \int \mathbf{1}_{\mathcal{V}}^{(\mathbf{a})}(i_v(t))d\mu_v^\times(t) & \text{if } S_0 = \{v\} \\ 0 & \text{if } S_0 \neq \{v\} \end{cases}$$

for all $\mathbf{a} \in \mathcal{J}$, $a \in \Gamma$ and places v of k. In turn it follows by formula (105) of §11.6 and the Γ-equivariance of ϵ that $\epsilon(\mathbf{a}) \in U$ for all $\mathbf{a} \in \mathcal{J}$. Finally, since by the preceding lemma and exactness of sequences (111) and (112) we may view ϵ as a $G(\infty)$-equivariant homomorphism $J(\infty) \to U$, in fact ϵ takes values in U^{ab} by Lemma 5.2. □

LEMMA 12.7. $\rho_{K/k}(\mathbf{a})\theta(0) = \mathbf{ord}\,\epsilon(\mathbf{a})$ *for all* $\mathbf{a} \in \mathcal{J}$.

Proof. For all $\mathbf{a} \in \mathcal{J}$, by (108) and (115) in the case $v = \infty$, we have

$$\mathrm{ord}_{\overline{\infty}}\,\epsilon(\mathbf{a}) = \int \Theta(i_\infty^\times(t), \mathbf{1}_{\mathcal{V}}^{(\mathbf{a})})d\mu_\infty^\times(t) = \sum_{n\in\mathbb{Z}} \Theta(\tau_\infty^n, \mathbf{1}_{\mathcal{V}}^{(\mathbf{a})}),$$

and so with $\mathbf{a}$ as above, the proof boils down to verifying the analytic identity

$$\rho_{K/k}(\mathbf{a})\theta(0) = \sum_{a\in\frac{\mathbb{A}^\times}{k^\times\mathcal{U}\langle\tau_\infty\rangle}} \left(\sum_{n\in\mathbb{Z}} \Theta(a^{-1}\tau_\infty^n, \mathbf{1}_{\mathcal{V}}^{(\mathbf{a})})\right)\rho_{K/k}(a) \tag{116}$$

relating partial zeta values to the theta symbol. Since the set of $\mathbf{a} \in \mathcal{J}$ satisfying the identity in question forms an ideal of $\mathbb{Z}[\Gamma]$, we may assume without loss of generality that $\mathbf{a} = 1 - q_{v_0}\cdot\tau_{v_0}$ for some $v_0 \in \overline{S}$. Let $\chi : G \to \mathbb{C}^\times$ be any character, and extend χ in $\mathbb{C}$-linear fashion to a ring homomorphism $\chi : \mathbb{C}[G] \to \mathbb{C}$. It is enough to prove the χ-version of (116), i. e., the relation obtained by applying χ to both sides of (116). Now we adapt to the present case the method presented in Tate's thesis for meromorphically continuing abelian L-functions. At least for $\Re(s) > 1$, when we

have absolute convergence and can freely exchange limit processes, we have

$$\begin{aligned}
& (1 - q_{v_0}^{1-s}\chi(F_{v_0})) \cdot \mu^\times \mathcal{U} \cdot \chi(\theta(s)) \\
= \quad & (1 - q_{v_0}^{1-s}\chi(F_{v_0})) \cdot \mu^\times \mathcal{U} \cdot \prod_{v \notin S_0} (1 - \chi(F_v) q_v^{-s})^{-1} \\
= \quad & (1 - q_{v_0}^{1-s}\chi(F_{v_0})) \cdot \int \mathbf{1}_{\mathcal{V}}(a)\chi(\rho_{K/k}(a))\|a\|^s d\mu^\times(a) \\
= \quad & \int \mathbf{1}_{\mathcal{V}}(a)(\chi(\rho_{K/k}(x))\|a\|^s - q_{v_0}\chi(\rho_{K/k}(\tau_{v_0}a))\|\tau_{v_0}a\|^s) d\mu^\times(a) \\
= \quad & \int (\mathbf{1}_{\mathcal{V}} - q_{v_0}\mathbf{1}_{\tau_{v_0}\mathcal{V}})(a) \cdot \chi(\rho_{K/k}(a))\|a\|^s d\mu^\times(a) \\
= \quad & \int_{\mathbb{A}^\times/k^\times} \Theta(a^{-1}, \mathbf{1}_{\mathcal{V}} - q_{v_0}\mathbf{1}_{\tau_{v_0}\mathcal{V}})\chi(\rho_{K/k}(a))\|a\|^s d\mu^\times(a).
\end{aligned}$$

The last integrand is constant on cosets of $(k^\times \mathcal{U})/k^\times$ and by Proposition 7.6 has compact support in $\mathbb{A}^\times/k^\times$. So the last integral defines an entire function of s. Now since $\mathcal{U} \cap k^\times = \{1\}$ we have $\mu(\mathcal{U}) = \mu((k^\times\mathcal{U})/k^\times)$. So by plugging in $s = 0$ at beginning and end of the calculation above and breaking the last integral down as a sum of integrals over cosets of $(k^\times\mathcal{U})/k^\times$ in $\mathbb{A}^\times/k^\times$, we recover the χ-version of (116) in the case $\mathbf{a} = 1 - q_{v_0} \cdot \tau_{v_0}$, as desired. $\square$

12.8 End of the proof

By the preceding two lemmas $\epsilon(\mathbf{a}_\infty)$ indeed has the properties specified in §12.1.5 characterizing $\epsilon(K/k, S)$. $\square$

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